AN INTRODUCTION

TO

GEOMETRY

UPON THE

ANALYTICAL PLAN.

BY

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PREFACE.

In the preparation of the present text-book for the use of beginners in Geometry, the needs of two different classes of students have been held in mind.

The primary design of the book, in its inception, was to furnish an Introduction to a text-book in Analytical Geometry, by the same author, already in great part prepared, and soon to be printed, provided the present book meet with a sufficiently favorable reception. It is thus intended that students in the higher schools — those aiming at an extended mathematical course, whether in college or elsewhere — shall find in this little book the opportunity to render that course *homogeneous from the beginning*, and at the same time *greatly to abbreviate* its initial stages. It frequently happens that a student, after painfully acquiring the method of Euclid only to find that it is to be laid aside for that of Descartes, experiences so much difficulty in the transition as never to become truly familiar with Algebraic Geometry or to appreciate the beauty of its processes. By using the first fifty-two pages of this text-book as an introduction to Analytical Geometry, the student finds his geometrical course, from its most elementary definition to the remotest application of the Calculus, only the orderly and harmonious development of a single subject according to a uniform plan.

The second class of students for whom it is hoped that this book may supply a need comprises those who aim at a practical use of elementary geometrical principles, and require only so much of the theory as will suffice to render this use intelligent. Such students will find the essential problems of land surveying and similar applications of Geometry brought in this text-book as near the threshold of the subject as their nature will admit.

As an Introduction to Analytical Geometry the first one hundred

and eleven sections are considered sufficient. The subsequent sections are appended with the view of rendering the present work complete in itself by embracing all the topics usually treated in Elementary Geometries and Plane Trigonometries, — and this largely for the use of the second class of students mentioned. In the author's Analytical Geometry those sections of the present book which follow Sect. 111 will either be incorporated in their proper places, or replaced by other methods of demonstration. The latter will be the course pursued, for instance, in regard to the Circle and all volumes bounded by curved surfaces; where the methods of Analytical Geometry proper, combined with the principle of the Center of Gravity, will furnish simpler demonstrations than could be brought within the compass of the present work.

Having thus defined the persons for whom and the objects for which this Introduction is especially intended, the author desires to say to any who may adopt it one urgent word in regard to the manner of its use. The ancient method of Euclid has unquestionably a great merit in this, that the actual forms dealt with are kept continually in the eye and mind of the student by its constant dependence upon diagrams. The diagrams employed in the demonstrations of this text-book are few, proofs from formulæ being always preferred for their generality. It is of the highest importance, therefore, that in every case of an algebraic demonstration THE STUDENT ILLUSTRATE AND TEST ITS APPLICATION BY DIAGRAMS DRAWN BY HIMSELF in as great a variety of form as possible. Unless this principle be applied throughout, the use of this book will entail failure from the lack of clear and intuitive ideas of the subjects presented, but by faithfully pursuing this course it is believed that all the advantages of the old and new methods may be combined.

It has been thought unwise to encumber this book with trigonometrical tables further than is necessary for illustration. When required for practical use, it is recommended to employ, in connection with the text of this Introduction, the volumes of tables by Prof. Loomis or Prof. Pierce.

F. H. L.

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CONTENTS.

In the notation of angles in this work the letters of the Greek alphabet are employed, which are as follows: —

α	Alpha.	ι	Iota.	ρ	Rho.
β	Beta.	$\varkappa$	Kappa.	σ	Sigma
γ	Gamma.	λ	Lambda.	τ	Tau.
δ	Delta.	μ	Mu.	υ	Upsilon.
ε	Epsilon.	ν	Nu.	φ	Phi.
ζ	Zeta.	ξ	Xi.	χ	Chi.
η	Eta.	o	Omicron.	ψ	Psi.
θ	Theta.	π	Pi.	ω	Omega.

MATHEMATICS is the science of measurement.

GEOMETRY is the mathematics of space.

Geometry, therefore, is founded on the axioms of quantity which are common to all mathematics, and with which the student has become familiar in Algebra, and on certain additional conceptions either involved in the idea of space, or inseparably associated with it. They resemble the proper axioms, or necessary truths, in this respect, that the mind accepts argument based upon them without questioning their validity, and usually without being aware that an admission has been made. Among these principles two may be instanced as of special importance: —

No point can be at once in two different directions from another point.

Any number of points may move in any direction without change of their relative directions or distances.

GEOMETRY.

PRIMARY DEFINITIONS.

1. A *point* is that which has no size, but only position.

Show that a dot merely *represents* a point.

2. If a point changes its position, or moves, its path is called a *line*.

Illustrated by the ordinary way of "drawing lines." The end of the pencil represents the moving point. Show that there is no *breadth*.

3. The moving point is called the *describing* or *generating* point, and the line is said to be *described* by it.

4. Any one of the successive positions occupied by the generating point is a *point on the line*.

5. A limited portion of a line is included between two points, and is called a *distance*. (See § 10.)

6. Lines are of three kinds, distinguished as follows:

(1.) When the generating point moves constantly in the *same direction*, the line it describes is called a *straight line*, or a *right line*.

(2.) When the generating point *constantly changes* its direction, the line described is called a *curved line*, or a *curve*.

(3.) When the generating point changes its direction only *at intervals*, the line described is called a *broken line*. (See Fig. 1.)

Illustrated by the manner in which each is drawn.

7. A broken line obviously is made up of several parts, each of which is a portion of a right line.

A curve may be considered as a broken line, in which these parts are extremely short.

8. Any right line may be considered as described in either of two opposite directions.

The generating point may be supposed to move in one direction to some indefinite extent, then to return on the same path, when it will, of course, describe the same line in an opposite direction.

Hence, a right line may be considered as described *from any point* on it, in both directions.

9. A right line must always be understood to be *unlimited* in extent.

Hence, we cannot speak of measuring *a right line*, but only distances upon it. A point is often designated by a letter of the alphabet, — then a right line is designated by naming any two points upon it, in the order in which the line is considered as described. The line BA, Fig. 1, is the same line as AB, but the directions indicated are opposite. A *distance* is also denoted by naming the points at each extremity; but, to avoid confusion, a vinculum will be used when this is intended. Thus, AB denotes a line passing through A and B; $\overline{AB}$ denotes the distance between these points.

10. *Distance* is thus far spoken of as a concrete quantity, — a portion of a line. The term is also used to denote an abstract number, — viz., the number of times any portion of a line contains that portion, or distance, which is used as a unit of measurement. In the latter sense, a distance may be denoted by any arithmetical or algebraic symbol for a quantity, and is subject (as concrete numbers are not) to every arithmetical or algebraic process, such as raising to powers, extracting roots, etc. The sense must be determined by the connection.

11. As a moving point describes a line, so a moving line generates a *surface*.

Any point of a line may be considered to move along some other line, carrying the first line with it. The place through which the line moves is then called a *surface*, the moving line is

called the *generating* line, or the *generatrix*, and the other line is called the *directrix*. The various kinds of surfaces depend on the form of these two lines, — whether they be straight, broken, or curved.

Illustrated by the use of carpenter's tools, etc. Show that there is length and breadth only.

12. When both the generatrix and directrix are *right lines*, and the former retains always the *same direction*, the surface generated is called a *plane*.

As the lines are unlimited (§ 9) so is the plane.

13. When a plane is generated, any point of the generating line describes a right line having *the same direction* as the directrix.

See Fig. 2, where AB is the path along which the point O of the generating line DC moves. Any other point on DC, as G, must move in the same direction as O, since DC is fixed in direction.

14. It thus appears that through any point of a plane may be drawn two right lines which shall be wholly in the plane, one having the same direction as the directrix, and the other having the direction of the generatrix; for the latter may be regarded as one of the successive positions which the generatrix assumes. But lines lying wholly within the plane may be drawn, not merely in these two directions, but in an indefinite number of different directions. In Fig. 2, suppose the plane to be described by the generatrix DC, whose point O moves along the directrix AB. Suppose that when the moving line has reached the position IK an obstacle intervenes at the point H, upon which the generatrix is forced to turn as on a pivot, the point O moving as before along AB. The motion of the generatrix no longer conforms to the requirement of § 12, but it will be seen that no point of the generatrix can reach any position which would not have been reached had the motion continued unchecked; hence all the varying positions of the generatrix lie wholly in one plane. Moreover, if any two points of a plane be given, one of them may be assumed as a

pivot upon which the generatrix may be conceived to swing, as above described, until it passes through the other point; hence,

A straight line joining two points of a plane lies wholly in the plane.

15. If, after the generatrix has swung to the position FL, the obstacle at H be removed, and the motion continued without subsequent change of the direction of the generatrix, then the same plane will be generated as if the original motion had continued without any obstacle. For since both generatrix and directrix are of unlimited extent, there is no point which would be reached by the generatrix in the one position which would not be reached in the other. Hence,

Any line of a plane may be taken as a generatrix.

Similarly,

Any line of a plane may be taken as a directrix.

But the directrix and generatrix must not extend in the same direction.

16. As a portion of a line is included between points, so a portion of a surface must be included within lines, or portions of them. When the including lines have such forms or positions that the included portion of the surface is finite in extent, it is called a *figure*.

17. As a moving line generates a surface, so any limited portion of a *plane* may move along a line not lying in that plane; and that which is thus produced is called a *volume*.

An entire plane thus moving along an unlimited line must pass through the whole of space. Hence a volume is a limited portion of space. As a figure is included within parts of lines, so a volume is included within parts of surfaces, which may be that by which it was generated (in two different positions), and those generated by the line or lines including the generating figure.

N. B. — In the following §§ whenever lines are mentioned, *right lines* are to be understood, unless the contrary

is stated; whatever is said of distances is to be understood of distances on right lines; and whatever is said of lines, points, etc., is to be understood of lines, points, etc., *in one plane*.

ANGLES.

18. An *angle* is the *difference in direction* of two lines, or the *amount of divergence* between two lines that pass through one point. (Fig. 3.)

To obtain the true idea of the magnitude of an angle, conceive two straight lines, of unlimited length, proceeding from a common point; let one of them be fixed in direction, and let the other be turned about the common point. When the two lines coincide, their angle is 0, and it increases as they separate. In the same way the *addition* of angles is illustrated, for this increase is a process of addition.

19. When two or more lines have the same direction, the angle between them is zero, and they are said to be *parallel*. (See Fig. 2, EP and AB.)

20. If, from a given point, two lines proceed in exactly opposite directions, their divergence is the greatest possible, and is taken as the standard for the measurement of angles.

An angle half as great as this is called a *right angle*.

Illustrated by the points of the compass.

The ninetieth part of a right angle is called a degree [°], the sixtieth part of a degree is called a minute [′], and the sixtieth part of a minute is called a second [″].

Hence, when a right line is considered as described (§ 8) from any point in it, in opposite directions, the angle between the two parts of the line on opposite sides of this point contains two right angles, or 180°.

Angles containing less than 90° are called *acute*, those containing more than 90°, *obtuse*, angles.

21. When the sum of two angles is equal to 180°, they are said to be *supplements* of each other; when their sum is equal to 90°, they are called *complements* of each other.

22. When two lines intersect (see Fig. 3), the four angles formed are named as follows: —

Those on the same side of one line, but on opposite sides of the other, are called *adjacent* angles, as α and β.

Those on opposite sides of both lines are called *opposite*, or *vertical*, angles, as α and α', or β and β'.

23. To investigate their relations, consider each line as described in both directions from the point of intersection, A. Then the sum of the angles α and β is evidently the angle between AE and AC, which is 180°. (See § 20.) Hence,

Adjacent angles are supplements of each other.

24. Both β and β' are adjacent to α, hence each is the supplement of α; $\therefore\ \alpha + \beta' = \alpha + \beta$, or $\beta' = \beta$. That is,

Vertical angles are equal to each other.

25. If $\alpha = 90°$, then $180° - \alpha = 90°$, or, a right angle is equal to its supplement. When one of the angles at the intersection of two lines is a right angle, the other three are also right angles, since two of them are supplements to it, and the third is vertical to it.

When the angles between two lines are right angles, the lines are said to be *perpendicular to each other*.

26. Since an angle is the difference in direction of two lines, any number of lines having the *same* direction will make equal angles with any line that crosses them.

Thus, in Fig. 4, if AB, CD, etc., are parallel, they will all make the same angles with IJ; and if KL is parallel to IJ, the angles it makes with AB, CD, etc., are equal to those IJ makes with them. α', α'', etc., each equals α, and β, β', β'', etc., each equals $180° - \alpha$. The same principle employed in §§ 23, 24, determines which angles are equal and which are supplements. The rule may be stated thus (compare § 22):

Those angles which lie on corresponding sides of both the lines which form them, or on opposite sides of both, are

equal; those which lie on corresponding sides of one, and opposite sides of the other, are supplements.

Thus, in Fig. 4, α and α' and α'', which all lie to the *right* of one and *above* the other of the lines which form them, are equal; α''' lies to the *left* of one, and *below* the other line forming it, hence this is equal to either of the former three; but α' and β' each lie to the *right* of one of their lines, while one is *above* and the other *below* the other line, hence these two angles are supplements.

27. When three lines intersect at different points, the angles between any two of them are distinguished as *interior* and *exterior* angles.

The interior angle is that produced when both the lines forming it are described from their intersection *toward* the third line.

The exterior angle is an adjacent angle to the interior, and therefore (§ 23) always its supplement.

Thus, in Fig. 5, when BA is described toward the line FE, and CD also toward that line, the difference between their directions is α which is the *interior* angle betweeen the lines. δ, or δ', which is produced when one of the same lines is described from their intersection toward FE and the other away from it, is the *exterior* angle. α', which is equal to the interior angle (§ 24), has no specific name.

28. When three lines intersect at different points, the exterior angle which two of them make with each other is equal to the sum of the interior angles which they make with the third line.

In Fig. 6, the exterior angle ζ adjacent to α is equal to the sum of the interior angles β and γ. For through the intersection A let a line pass parallel to PQ, and dividing the angle ζ into two parts δ and ε. Now, $\delta = \beta$ and $\gamma = \varepsilon$ (§ 26), hence $\zeta = \delta + \varepsilon = \beta + \gamma$.

29. When three lines intersect at different points, the sum of the interior angles is equal to 180°.

For $\alpha + \zeta = 180°$ (§ 23), but $\zeta = \beta + \gamma$ (§ 28) $\therefore \alpha + \beta + \gamma = 180°$.

30. If two of the lines are perpendicular to each other, the angle between them is equal to 90°, hence the sum of the other two angles must be equal to (180° — 90°) or 90°. That is: —

When a line intersects two other lines that are perpendicular to each other, the interior angles which it makes with them are complements.

31. Two lines perpendicular to the same line are parallel.

In Fig. 7 let AB and CD each be perpendicular to BD, then are they parallel. For, if not, suppose some other line, as ED, passing through D, to be parallel to AB; then (§ 26) the angle which it makes with BD is equal to α, that is, to 90°. But, by supposition, the angle made by CD with BD is 90°, hence these two angles are equal, i. e., the part to the whole, an absurdity; therefore CD is parallel to AB.

Hence, only one perpendicular can be drawn from a point to a line.

For if two lines, passing through the same point, *have the same direction*, obviously they coincide in one line.

32. It is of special importance that the pupil give definiteness to his own conceptions by drawing figures to illustrate the principles learned, as far as possible. For this purpose he needs at this stage of his progress no other instruments than he can make for himself, in accordance with the foregoing principles.

Let a piece of tolerably stiff paper be folded once; the edge of the fold will represent a straight line. Now fold it again, taking care that the edge of the former fold is turned accurately upon itself. The angle where the two folds meet will be a right angle. For the two angles on each side of the second fold are equal, since the folding has made them coincide, and together they make up the divergence of the two parts of a straight line. (§ 20.)

With this instrument (which we will call a square) the following problems may be solved: —

A. To draw a line perpendicular to a given line, from a given point upon it. Place the square at the given point, so that one edge falls on the given line, the other edge will serve as a ruler by which to draw the perpendicular.

B. To draw a line perpendicular to a given line, from a given point without it. Place the square so that one edge coincides with the given line, and slide it upon the latter, until the other edge passes through the given point.

C. To draw a line parallel to a given line, through a given point. Draw any line perpendicular to the given line, then through the given point draw a line perpendicular to this. (§ 31.)

By means of these rules, let the pupil construct illustrative figures for all the following demonstrations, varying the form as much as possible from that given on the chart.

For some subsequent uses, it will be convenient to have a scale of equal parts drawn on each edge of the square. These should be copied from some standard, as a foot-rule, so that all may be alike. A fourth or a fifth of an inch is a convenient unit, if the drawing is to be on paper. The right angle of the square should be marked 0, and the equal divisions counted in each direction from this point.

SYSTEMS OF LINES.

33. A *system of lines* consists of any number of lines having some mutual relation. Thus, if a number of lines all have the same direction, they are called a system of parallels; if they all pass through one point they are called a system of converging lines, or of convergents, and the common point is called the point of convergence.

34. If two or more lines, forming a system of convergents, are intersected by another line, that part of any one of the convergents included between the intersecting line and the point of convergence is called the *intercept* of the intersecting line on that convergent.

Fig. 8 represents a system of three convergents, AB, AC and AD. $\overline{AG}$ is the intercept of MN on AD, $\overline{AF}$ is its intercept on AC, etc.

35. The parts of two parallels, included between two other parallels, are equal.

Take a line (EF, Fig. 9,) of one of the systems as the generating line of the plane in which both are situated (§ 15), and a line, AB, of the other system for the directrix. The line CD will then be described by the point J (§ 13). Now, GH has the same direction with EF; therefore, if EF moves to the right (retaining its direction), when the point I reaches the point K the two lines will coincide, and the point J will fall on the point L. Therefore the distances $\overline{IJ}$ and $\overline{KL}$ must be equal.

36. Let any number of points be taken at equal distances on the same right line; through these points let there pass two different systems of parallels, intersecting each other;

then, a right line passing through any one of their intersections, and parallel to the first right line, will pass through the next consecutive intersection.

See Fig. 10, where A, B, C, etc., are at equal distances on the line AE. Through these pass the parallels AO, BL, CM, etc., and also the parallels AP, BQ, CR, etc. H, K, etc., are the intersections of the two systems. Now, a line passing through H, and parallel to AE, will pass through the next intersection, K. For, let the point in which such a line meets the line CR be denoted by X, and that in which it meets BL be denoted by Y. Then, since $\overline{HY}$ and $\overline{AB}$ are parts of parallels, included between the parallels AO, BL, they are equal. Also $\overline{HX}$ and $\overline{BC}$ are equal, being parts of parallels included between the parallels BQ and CR. But $\overline{BC}$ was taken equal to $\overline{AB}$ $\therefore$ $\overline{HX} = \overline{HY}$, that is, the points X and Y, in the same direction from H, are at the same distance from it, or they are the same point, and therefore identical with K, since that is the only point situated on both the lines CR and BL.

[It is here assumed that two lines can have only one point in common. This is an obvious consequence of the definition of a right line. For to suppose that the two lines CR and BL, passing through K in two different directions, could meet at some other point, would involve the absurdity that this second point could be at once in two different directions from K.]

If the line passing through H, and parallel to AE, passes through K, it will for the same reason pass through I, etc., etc.

37. If any point be taken on one of two intersecting lines, its distance from the other line, measured in any given direction, will be in a constant ratio to its distance from the intersection of the two lines.

Thus, in Fig. 11, *a*, $\overline{DE}$ is the distance of a point, E, from the line AB, measured in a given direction, PQ, and AE is the distance of the same point from the intersection A; and it is to be proved that the ratio $\dfrac{\overline{DE}}{\overline{AE}}$ is *constant*, i.e., that it is the same wherever on the line AC the point E may be taken.

On this line AC let a number of points (F, G, H, W, etc.) be taken so that the distances $\overline{AF}$, $\overline{FG}$, $\overline{GH}$, etc., shall be equal. (See Fig. 11, *b*.) Through these points let the system of parallels IF, JG, KH, etc., pass, in the given direction of measurement, PQ. Also through the same points let lines FM, GN, etc., pass parallel to AB. Through the points I, J, K, etc., where the line AB intersects the former system, let lines pass parallel to AC. They will pass through the intersections S, T, U, Y, etc. (§ 36.)

Now, since $\overline{IF}$, $\overline{JS}$, are parts of parallels included between the parallels AB and FM, they are equal. And since $\overline{IF}$, $\overline{SG}$ are parts of parallels included between the parallels AC and IV, they also are equal, In the same way, $\overline{KU}$, $\overline{UT}$, $\overline{TH}$, $\overline{LY}$, $\overline{YX}$. $\overline{XV}$, $\overline{VW}$, etc., may each be proved equal to $\overline{IF}$. Therefore $\overline{JG} = 2 \times \overline{IF}$, $\overline{KH} = 3 \times \overline{IF}$, $\overline{LW} = 4 \times \overline{IF}$, etc. ; $\therefore$ the ratio of the distances from the line AB and the point A is the same for each of these points F, G, H, etc., viz., $\frac{\overline{IF}}{\overline{AF}}$. For $\frac{\overline{JG}}{\overline{AG}} = \frac{2 \times \overline{IF}}{2 \times \overline{AF}} = \frac{\overline{IF}}{\overline{AF}}$; $\frac{\overline{KH}}{\overline{AH}} = \frac{3 \times \overline{IF}}{3 \times \overline{AF}} = \frac{\overline{IF}}{\overline{AF}}$; etc. Now, the equal distances $\overline{AF}$, $\overline{FG}$, etc., between the successive points A, F, G, etc., might have been taken at any value either greater or less than the one actually employed. Hence, they may be taken at such a value that one of the points, F, G, etc., will fall on any point on the line AC that may be assigned, or we may regard them as so small that these points should coincide with *all* the successive points of the line, yet the demonstration would not be impaired. Hence, what has been proved of these points must be true of every point of the line AC, viz., that its distance from the given line, AB, measured in the given direction, PQ, has a *fixed ratio* to its distance from the intersection A.

38. For any given direction of measurement, it is evident that the magnitude of this ratio will depend simply upon the angle between the two given lines. For at any given distance from the intersection of two lines, their distance apart will depend upon the amount of their divergence. This is expressed in the words, "the ratio is a function of the angle."

FUNCTIONS OF ANGLES.

39. When the value of one quantity depends upon the value of another, the former quantity is called a *function* of the latter. Thus, the quantity $(10\,\mathbf{a}^2 + 3)$ is a function of $\mathbf{a}$. There are certain functions of *angles* which are used with great advantage in calculations where the use of the number directly expressing the magnitude of the angle (in degrees, etc.) would be extremely inconvenient, or even impossible. Those functions in most common use are the sine, tangent, cosine, and cotangent of angles.

40. When two lines which form an angle are intersected by a third line which is perpendicular to one of them, the ratio of that part of the third line which is included between the other two to its intercept on the line to which it is perpendicular, is called the *tangent* of the angle.

41. The ratio of the same part of the third line to its intercept on the line to which it is *not* perpendicular, is called the *sine* of the angle.

Thus, in Fig. 12, when the lines AM, AN, including the angle a, are intersected by the line BC perpendicular to AM, the ratio $\dfrac{\overline{\mathrm{KH}}}{\overline{\mathrm{AK}}}$ is the *tangent* of the angle a. This ratio is the same at whatever point the line BC may meet AM, provided only it be perpendicular to AM (§ 37), for that fact fixes the direction in which the distance $\overline{\mathrm{KH}}$ is measured as one differing 90° from that of AM. Also (§ 38), this ratio is a function of the angle a.

The ratio $\dfrac{\overline{\mathrm{KH}}}{\overline{\mathrm{AH}}}$ is the *sine* of the angle a. It is the same at whatever point BC may meet AN (§ 37), and is a function of the angle a. (§ 38.)

42. The *cosine* of an angle is the sine of the complement of that angle. The *cotangent* of an angle is the tangent of the complement of that angle.

The prefix "co" is here an abbreviation of "complement."

43. For use in formulæ, etc., the names of these four functions are abbreviated into *tan*, *sin*, *cos*, and *cot*. Thus, tan α is read "the tangent of α," sin 25° is read "sine of twenty-five degrees." When one of these abbreviations is followed by the sign ($^-$) the *angle* is indicated in terms of its function. Thus, $\cos^- \frac{1}{2}$ is "the angle whose cosine is one-half;" $\cot^- 3$, "the angle whose cotangent is three."

44. In Fig. 13, if AB and AD are perpendicular to each other, the angle β is the complement of α. Through any point, P, on the line AC let a line pass parallel to AD and another parallel to AB. The former will be perpendicular to AB and the latter to AD. (§ 26.) Then $\frac{\overline{\text{MP}}}{\overline{\text{AM}}} = \tan\alpha$, $\frac{\overline{\text{MP}}}{\overline{\text{AP}}} = \sin\alpha$, $\frac{\overline{\text{NP}}}{\overline{\text{AN}}} = \tan\beta = \cot\alpha$, and $\frac{\overline{\text{NP}}}{\overline{\text{AP}}} = \sin\beta = \cos\alpha$. Now, $\overline{\text{AM}} = \overline{\text{NP}}$. (§ 35.) $\therefore \frac{\overline{\text{NP}}}{\overline{\text{AP}}} = \frac{\overline{\text{AM}}}{\overline{\text{AP}}}$, the ratio of the intercepts of PM on AB and AC. Hence we may give a new definition of the cosine, as follows:

When two lines, which form an angle, are met by a third line, which is perpendicular to one of them, the ratio of its intercept on that line to its intercept on the other is called the *cosine* of the angle.

45. Since $\overline{\text{AM}} = \overline{\text{NP}}$ and $\overline{\text{AN}} = \overline{\text{MP}}$ (§ 35), $\cot\alpha = \tan\beta = \frac{\overline{\text{NP}}}{\overline{\text{AN}}} = \frac{\overline{\text{AM}}}{\overline{\text{MP}}}$, the ratio of the intercept $\overline{\text{AM}}$ to the part $\overline{\text{MP}}$ of the intersecting perpendicular. Hence we may give the following definition of the cotangent:

When two lines, which form an angle, are met by a third line, which is perpendicular to one of them, the ratio of its intercept on that line to that part of the intersecting perpendicular included between the first two lines is called the *cotangent* of the angle.

These two definitions may be applied, as well as the analogous ones of §§ 40, 41, to Fig. 12, where we have $\frac{\overline{KH}}{\overline{AH}} = \sin \alpha$, $\frac{\overline{KH}}{\overline{AK}} = \tan \alpha$, $\frac{\overline{AK}}{\overline{AH}} = \cos \alpha$, $\frac{\overline{AK}}{\overline{KH}} = \cot \alpha$.

46. All these definitions are applicable, not only to angles less than 90°, but to angles from 90° to 180° as well. In the latter case, however, one of the lines forming the angle must be considered as extending beyond the point of intersection. Thus, in Fig. 14, where β is the angle between AM and AN, $\frac{\overline{KH}}{\overline{AH}} = \sin \beta$, $\frac{\overline{KH}}{\overline{AK}} = \tan \beta$, $\frac{\overline{AK}}{\overline{AH}} = \cos \beta$, and $\frac{\overline{AK}}{\overline{KH}} = \cot \beta$.

Moreover, though, as we have already seen (§ 20), an angle of 180° is the greatest possible divergence between two lines, yet it is very common to speak of angles of from 180° to 360°, or even of angles of any number of degrees whatever. This usage is the natural and even necessary result of the simple process of the addition of angles. Thus, in Fig. 15, if the angle $\alpha = 110°$ and $\beta = 100°$, it is natural to speak of their sum as an angle γ, equal to 210°. And here, exactly as in the former cases, $\frac{\overline{KH}}{\overline{AH}} = \sin \gamma$, $\frac{\overline{KH}}{\overline{AK}} = \tan \gamma$, $\frac{\overline{AK}}{\overline{AH}} = \cos \gamma$, and $\frac{\overline{AK}}{\overline{KH}} = \cot \gamma$.

The definitions of § 41 are no less applicable to these cases than those of §§ 44, 45; as the student will see when he reaches the discussion of negative angles.

47. When two lines perpendicular to each other are intersected by another line, the square of that part of the latter included between them is equal to the sum of the squares of its intercepts on the first two lines.

In Fig. 16, let there pass through the intersection of FE and CD (which are perpendicular to each other) a line perpendicular to AB, and cutting it in K, so as to divide the distance $\overline{GH}$ or s

into two parts, $\overline{GK}$ or $\mathbf{p}$, and $\overline{KH}$ or $\mathbf{q}$. Let the distances $\overline{GO}$ and $\overline{HO}$ be called $\mathbf{m}$ and $\mathbf{n}$. Then it is to be proved that

$$\mathbf{s}^2 = \mathbf{m}^2 + \mathbf{n}^2. \quad \text{(See § 10.)}$$

$$\operatorname{Cos} \alpha = \frac{\mathbf{p}}{\mathbf{m}}, \text{ and } \cos \alpha = \frac{\mathbf{m}}{\mathbf{s}}, \therefore \frac{\mathbf{p}}{\mathbf{m}} = \frac{\mathbf{m}}{\mathbf{s}}, \text{ and } \mathbf{ps} = \mathbf{m}^2. \quad (1.)$$

In the same way,—

$$\cos \beta = \frac{\mathbf{q}}{\mathbf{n}}, \text{ and } \cos \beta = \frac{\mathbf{n}}{\mathbf{s}}, \therefore \frac{\mathbf{q}}{\mathbf{n}}, = \frac{\mathbf{n}}{\mathbf{s}}, \text{ and } \mathbf{qs} = \mathbf{n}^2. \quad (2.)$$

Adding (1) and (2) we have

$$\mathbf{ps} + \mathbf{qs} = \mathbf{m}^2 + \mathbf{n}^2,$$

or

$$\mathbf{s}(\mathbf{p} + \mathbf{q}) = \mathbf{m}^2 + \mathbf{n}^2.$$

But $\quad \mathbf{p} + \mathbf{q} = \mathbf{s}, \quad \therefore \quad \mathbf{s}^2 = \mathbf{m}^2 + \mathbf{n}^2.$

48. Dividing this equation through by $\mathbf{s}^2$, we have

$$1 = \frac{\mathbf{m}^2}{\mathbf{s}^2} + \frac{\mathbf{n}^2}{\mathbf{s}^2}, \quad \text{or} \quad \left(\frac{\mathbf{n}}{\mathbf{s}}\right)^2 + \left(\frac{\mathbf{m}}{\mathbf{s}}\right)^2 = 1.$$

But $\quad \frac{\mathbf{n}}{\mathbf{s}} = \sin \alpha$ and $\frac{\mathbf{m}}{\mathbf{s}} = \cos \alpha, \quad \therefore \quad \sin^2 \alpha + \cos^2 \alpha = 1.$

(These expressions are read, "the square of the sine of α," etc.)

Hence the sum of the squares of the sine and cosine of any angle is equal to unity.

49. The ratio $\frac{\overline{KH}}{\overline{AK}}$ (Fig. 12), is equal to $\frac{\overline{KH}}{\overline{AH}} \div \frac{\overline{AK}}{\overline{AH}}$; or in general terms: —

The tangent of an angle is equal to the sine divided by the cosine; or $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$.

Hence $\quad \sin \alpha = \cos \alpha \tan \alpha$, and $\cos \alpha = \frac{\sin \alpha}{\tan \alpha}$.

50. The ratio $\quad \frac{\overline{AK}}{\overline{KH}} = \frac{\overline{AK}}{\overline{AH}} \div \frac{\overline{KH}}{\overline{AH}}$; or

The cotangent is equal to the cosine divided by the sine; or $\cot \alpha = \frac{\cos \alpha}{\sin \alpha}$.

Hence $\cos a = \sin a \cot a$, and $\sin a = \frac{\cos a}{\cot a}$.

These results follow from § 49, for, if $\tan a = \frac{\sin a}{\cos a}$,

then $\tan (90° — a) = \frac{\sin (90° — a)}{\cos (90° — a)}$; or $\cot a = \frac{\cos a}{\sin a}$.

51. The ratio $\frac{\overline{AK}}{\overline{KH}} = 1 \div \frac{\overline{KH}}{\overline{AK}}$; or,

The cotangent is the reciprocal of the tangent;

i. e., $$\cot a = \frac{1}{\tan a},$$

whence $\tan a \cot a = 1$, and $\tan a = \frac{1}{\cot a}$.

This may be obtained by comparing §§ 49, 50; for if

$\tan a = \frac{\sin a}{\cos a}$, and $\cot a = \frac{\cos a}{\sin a}$, evidently $\cot a = \frac{1}{\tan a}$.

52. By means of his square, with graduated edges, the pupil can now solve the following problems:

A. To measure the sine or cosine of a given angle.

Selecting a convenient number of units, such as 10, lay off the distance on one of the lines forming the given angle, and from the point thus fixed draw a perpendicular to the other line. Then measure the distances named in the definitions (§§ 41, 44), and divide by the assumed number of units.

B. To measure the tangent of a given angle.

Select a convenient number of units, as before, and lay it off from the intersection of one of the lines, then draw a perpendicular to that line at the point thus fixed, and measure the part included between the given lines. Divide this length by the selected number of units.

C. To measure the cotangent of an angle, the most convenient way will be to measure its tangent and take the reciprocal.

D. To construct, at a given point in a given line, an angle having a given sine or cosine.

Lay off on the given line from the given point, A, a distance, $\overline{AB}$, equal to the denominator of the given ratio. Take on one edge of the square a distance equal to the numerator of the same, and if the ratio be a sine, fix this point of the square at the point B, but if the given function be a cosine, at the point A, and turn the square until the remaining edge passes through the other point. Then draw a line through A, making the required angle. See Fig. 17, where $a = \sin^{-} \frac{1}{2}$.

If the distances indicated by the terms of the fraction are inconveniently large or small, the fraction should first be reduced, by dividing or multiplying both numerator and denominator alike.

E. To construct an angle whose tangent or cotangent is given.

Lay off on the line, as before, a distance equal to the denominator of the ratio (reduced, if convenient), if the ratio be a tangent, but if the cotangent is given, lay off the numerator. At the point thus fixed erect a perpendicular equal to the remaining term of the ratio, and connect its extremity with the given point.

Examples.

1. Construct the angle whose sine is $\frac{1}{2}$, that whose tangent is 2, one whose cosine is $\frac{2}{3}$, and one whose cotangent is $\frac{4}{5}$.

2. Construct $\cos^{-}\frac{1}{2}$, $\tan^{-}\frac{3}{4}$, $\sin^{-}\frac{2}{3}$, $\cot^{-}3$, $\sin^{-}\frac{3}{5}$, $\cos^{-}\frac{2}{5}$, $\tan^{-}\frac{1}{2}$, $\cos^{-}\frac{5}{13}$, $\cot^{-}\frac{5}{12}$.

The teacher may multiply these examples indefinitely, and should provide examples for the converse rules, of measuring the functions of given angles.

3. Compute the cosine of the angle whose sine is $\frac{3}{5}$, the tangent of the angle whose cotangent is $\frac{2}{3}$.

4. Compute all the other functions of $\cos^{-}\frac{3}{5}$, of $\tan^{-}\frac{5}{12}$.

In the last example we easily find $\cot = \frac{12}{5}$. To find the cosine we have from § 49, $\sin a = \cos a \tan a$, $\therefore \sin^2 a = \cos^2 a \tan^2 a$, that is, in this case (since $\tan a = \frac{5}{12}$, and hence $\tan^2 a = \frac{25}{144}$), $\sin^2 a = \frac{25}{144} \cos^2 a$. Now, let x represent $\cos^2 a$, then $\frac{25}{144} x = \sin^2 a$, and (§ 48) $x + \frac{25}{144} x = 1$, or $\frac{169}{144} x = 1$, whence $x = \frac{144}{169}$, the square of the cosine of a. Hence, the cosine of $a = \frac{12}{13}$, and, since $\sin a = \cos a \tan a$, $\sin a = \frac{12}{13} \cdot \frac{5}{12} = \frac{5}{13}$.

5. Compute the other functions of $\cot^{-}\frac{4}{3}$, of $\cos^{-}\frac{5}{13}$.

After performing examples 3, 4, 5, construct all the angles from the given functions, and measure the required functions to prove the work.

6. On each of two lines, making right angles with each other, is measured from their intersection a distance of 100 feet. What is the distance between the two points thus fixed? (See § 47.) Answer: 141.42 +.

7. If a staff 4 feet long, held perpendicularly, casts a shadow 3 feet long, what is the distance from the top of the staff to the end of the shadow?

8. My brother's house is due north from mine, and due west from my cousin's. My house is $\frac{5}{8}$ mile distant from my cousin's, and $\frac{1}{2}$ mile from my brother's. What is the distance from my brother's house to my cousin's?

9. A may-pole 32 feet high, standing on a level plain, is broken partly through, so that the top rests on the ground 16 feet from the foot of the pole. How far above the ground is it broken?

[Let $x =$ the length of the part broken off, and $y =$ the length of the remainder (the distance required). Then we may obtain $(x + y)(x - y) = 256$, whence, as $x + y$ is known, y is readily found. Answer: 12.]

10. From the top of the mast of a sail-boat a rope is stretched to a point on the deck 9 feet distant from the foot of the mast. The rope is three feet longer than the mast. How tall is the mast?

11. The village A is due north from B, and B is due west from C, to which straight roads extend from both A and B. Two travelers, the one from A, the other from B, met at C; and, on comparing the distances travelled, it was found that one had come five miles further than the other, and that the entire distance traveled by both was three times the distance, in a direct line, between the starting points. How far had each traveled?

12. A township, 1,920 rods square, is divided into sections by lines parallel to the boundaries, at distances of 320 rods. Required, the distances between points in the township to be located by the teacher.

53. If a line meet a system of convergents, the sines of the angles which it makes with any two of them are reciprocally proportional to its intercepts upon them.

Thus, let PM, PN, and PO (Fig. 18), be a system of convergents intersected by the line AX, which makes, with PM, the angle α, with PN the angle β, and with PO the angle γ. Then will these proportions be true, viz.,

$$\text{Sin}\,\alpha : \sin\beta :: \overline{PN} : \overline{PM},$$

$$\sin\alpha : \sin\gamma :: \overline{PO} : \overline{PM},$$

and

$$\sin\beta : \sin\gamma :: \overline{PO} : \overline{PN}.$$

For, through P, the point of convergence, let the line PR pass perpendicular to AX, and meeting it at R. Then,

$$\sin\alpha = \frac{\overline{PR}}{\overline{PM}},\ \sin\beta = \frac{\overline{PR}}{\overline{PN}},\ \text{and}\ \sin\gamma = \frac{\overline{PR}}{\overline{PO}}.$$

Hence, $\overline{PR} = \overline{PM} \sin \alpha = \overline{PN} \sin \beta = \overline{PO} \sin \gamma$.

Now, from the equation

$$\overline{PM} \sin \alpha = \overline{PN} \sin \beta,$$

may be derived $\sin \alpha : \sin \beta :: \overline{PN} : \overline{PM}$;

from $\overline{PM} \sin \alpha = \overline{PO} \sin \gamma$,

comes $\sin \alpha : \sin \gamma :: \overline{PO} : \overline{PM}$,

and from the equation

$$\overline{PN} \sin \beta = \overline{PO} \sin \gamma,$$

we have $\sin \beta : \sin \gamma :: \overline{PO} : \overline{PN}$.

The pupil may here urge the objection that the theorem is not sufficiently definite. For "the angle which AX makes with PM" might be the angle α' as well as α. Which of two such angles is intended?

Answer: *Either;* for the *sines* of α and α' are identical, viz., $\dfrac{\overline{PR}}{\overline{PM}}$, although the angles are different. (§ 46.) Hence the statement should not be more explicit.

That sin α and sin α' are identical in algebraic sign as well as in numerical value will appear hereafter. (§ 89.)

DETERMINATION OF POINTS IN A PLANE.

54. The position of a point in a plane is determined when its *distance* and *direction* from a known point are given.

This is the most obvious and natural way of describing the position of a point, and examples of it in daily use are abundant.

55. *Direction* from the known point is given by comparison with a known direction.

Thus, (Fig. 19,) the position of P is determined if we know that it is at a distance of one inch from the known point, A, on a line, AQ, which makes an angle of 30° above the line AX, whose direction is known.

56. The quantities which must be given, in order to determine the position of a point, are called the *coördinates* of that point.

The coördinates of P, (Fig. 19,) are the distance $\mathbf{r} = 1$ inch, and the angle $\theta = 30°$.

57. This method of determining the position of a point, where the coördinates are a distance and an angle, is called

The Method of Polar Coördinates.

58. The known point, A, is called the *pole*, and the line AX, passing through the pole in the known direction, is the *initial line*.

59. The distance $\mathbf{r}$ is called the *radius vector*, and the angle θ, the *vectorial angle*. These two are the *polar coördinates*.

60. By general consent, the initial line is always described *from* the pole, *toward the right*. The radius vector is also always considered as described *from the pole*, and the vectorial angle is measured continuously around the pole from right to left, — i. e., in a way contrary to the motion of the hands of a watch.

Thus, (Fig. 20,) if the line AB makes an angle of 45° with AX, and AC makes an angle of 45° with AB, and so on; the vectorial angle for any point on the line AB is 45°; on the line AC, 90°; on AD, 135°; on AE, 180°; on AF, 225°; on AG, 270°; and on AH, 315°. The vectorial angle for points on the line AX may be considered either 0° or 360°. We may continue, and regard the line AB as corresponding to a vectorial angle of $360° + 45°$, or 405°, as well as of 45°, while the value of θ for a point on AC is either 90° or 450°, etc., and so the circuit may be repeated as many times as we please. That is,

The direction of the radius vector is not altered when the vectorial angle is either increased or diminished by 360°, or any multiple of that quantity by a whole number.

61. *Problem.* — To locate the point whose coördinates are $\mathbf{r} = \mathbf{m}$, $\theta = (\alpha + \beta)$; also the point whose coördinates are $\mathbf{r} = \mathbf{m}$, $\theta = (\alpha - \beta)$.

For the solution of either part of this problem it is obvious that

two lines must be found passing through the pole, the one making an angle equal to α above the initial line, the other making an angle of β with the former. But in the first case, since the *sum* of two angles is required, the angle β must be taken on the upper or left-hand side of the first line drawn, while in the second case it must be taken on the lower or right-hand side of that line, so that the inclination of the second line shall be *less* than that of the first by β. In either case, a distance equal to $\mathbf{m}$ must be measured from the pole on the second line, and its extremity will locate the required point.

From this example it will be seen that the negative sign indicates a *reversed direction of measurement*, for when $\theta = \alpha + \beta$ we measure both α and β from right to left, but when $\theta = \alpha - \beta$, the latter is measured from left to right. Obviously, if β exceeds α in numerical value, the line on which the radius vector is to be measured will fall *below* the initial line, but in this case the value of the expression $\alpha - \beta$ becomes negative, hence the formula $\theta = -\gamma$ indicates that a vectorial angle equal to γ is to be measured *below* the initial line. In all cases

Negative angles are measured from left to right, as positive angles are measured from right to left.

62. Any negative angle may be rendered positive by adding to it 360° a sufficient number of times, and any positive angle greater than 360° may be made less than 360° by subtracting that quantity or a sufficiently large multiple of it, and since these operations do not affect the direction of the radius vector (§ 60), it follows that

Any point whose vectorial angle is negative, or positive and greater than 360°, may be denoted by a positive vectorial angle less than 360°, without change of the radius vector.

63. *Problem.*—To locate the point whose coördinates are $\theta = \alpha$, $\mathbf{r} = \mathbf{m} + \mathbf{n}$; also the point whose coördinates are $\theta = \alpha$, $\mathbf{r} = \mathbf{m} - \mathbf{n}$.

Having found, as before, a line making an angle equal to α above the initial line, we measure upon it from the pole a distance equal to $\mathbf{m}$. Both the points in question are upon this line, and at a

distance of **n** from the point already located, but it is obvious that when the radius vector is the *sum* of **m** and **n**, the measurement of **n** should be continued in the same direction in which **m** was measured; but when the *difference* is sought, the measurement should be in the contrary direction. If **n** is numerically greater than **m**, the point will fall beyond the pole. Hence as before, the negative sign indicates a reversed direction of measurement; and

Negative radii vectores are measured in a direction contrary to that indicated by the vectorial angle.

64. The principle illustrated in §§ 61, 63, is clearly a universal one, and it will hereafter be assumed that whenever measurement in a determinate direction is considered as affording a positive result, that which is measured in the opposite direction is of necessity regarded as negative.

65. The difference between two opposite directions is 180°; hence a change of sign in the radius vector produces the same effect as an increase or decrease of the vectorial angle by 180°; and if both these operations are performed at once they must neutralize each other. Therefore,

Any point whose radius vector is negative may be denoted by a positive radius vector of the same length, if the vectorial angle be increased or diminished by 180°.

Examples.

Locate the points whose coördinates are as follows. $\mathbf{r} = 4$, $\theta = 90°$; $\mathbf{r} = 3$, $\theta = \sin^{-}\frac{1}{2}$; $\mathbf{r} = 5$, $\theta = \tan^{-}2$; $\mathbf{r} = \frac{4}{3}$, $\theta = \cot^{-}\frac{2}{3}$; $\mathbf{r} = \frac{5}{2}$, $\theta = \cos^{-}\frac{4}{5}$, etc., etc.

Reduce the following coördinates to positive ones, in which $\theta < 360°$, and locate the points. $\mathbf{r} = 5$, $\theta = -90°$; $\mathbf{r} = -3$, $\theta = 270°$; $\mathbf{r} = 3$, $\theta = -270°$; $\mathbf{r} = -3$, $\theta = -450°$, etc.

66. While for some geometrical uses this "method of polar coördinates" is very serviceable, and, indeed, essential, in general its application is inconvenient, because the use of both distances and angles is necessary throughout. Hence another

method of determining the position of a point is more frequently employed, called, from the name of its inventor, Descartes,

The Cartesian Method of Coördinates.

In this method all the quantities employed are distances, and the position of a point is determined by reference to *two intersecting right lines*, the position of which is known. These lines (AX and AY, Fig. 21,) are called the *axes*, and the point (A) at which they intersect is called the *origin*.

67. The coördinates of a point in the Cartesian method, are its distances from the two axes, the distance from either axis being measured on a line parallel to the other. Thus, RP, QP, are the coördinates of the point P. Since RP = AQ and QP = AR (§ 35) either coördinate may be measured *on one of the axes*, from the origin to the intersection of a parallel to the other axis.

Let the student notice that the coördinates are always measured *from* the axes, or *from* the origin, *never toward them.*

68. One of the axes is always considered as passing through the origin *toward the right*, and the coördinate measured on this axis, or parallel to it, is called the *abscissa*, the axis itself being therefore called the axis of abscissas. Thus, AX is the axis of abscissas, and AQ or RP is the abscissa of P.

69. The coördinate measured on the other axis, or parallel to it (AR or QP) is called the *ordinate*, and this axis (AY) is named the axis of ordinates.

70. Abscissas are usually denoted by the letter **x**, and ordinates by the letter **y**; hence the axis of abscissas is frequently called the axis of X, and the axis of ordinates, the axis of Y.

71. Abscissas measured toward the right are considered positive, hence, (§ 64) those measured to the left must be considered negative. Ordinates measured upward are positive, hence those measured downward are negative.

Thus, in Fig. 21, the coördinates of P are both positive; those of P″ are both negative; for P′ the ordinate is positive, and the abscissa negative; and for P‴ the abscissa is positive, but the ordinate is negative.

It is obvious that all points cannot be denoted in the Cartesian as in the polar method, by the use of positive coördinates only.

72. The two axes divide the plane in which they are situated into four parts, of which that situated above the axis of X and to the right of the axis of Y is called the *first angle;* that above the axis of X and to the left of the axis of Y, the *second angle;* that below the axis of X and to the left of the axis of Y, the *third angle;* and that below the axis of X and to the right of the axis of Y, the *fourth angle.*

Hence, for points in the first angle, $\mathbf{x}$ is $+$ and $\mathbf{y}$ is $+$; in the second angle, $\mathbf{x}$ is $-$ and $\mathbf{y}$ is $+$; in the third angle, $\mathbf{x}$ is $-$ and $\mathbf{y}$ is $-$; in the fourth angle, $\mathbf{x}$ is $+$ and $\mathbf{y}$ is $-$.

73. The angle between AX and AY, denoted by the letter ω, is of course a known angle, since the position of both axes is known. It may have any value from 0° to 180°. When it is a right angle, the Cartesian method becomes *the method of rectangular coördinates*, otherwise it is *the method of oblique coördinates.*

For most geometrical uses, the processes of calculation are much simpler when rectangular coördinates are used than when they are oblique. Hence, in the following §§ the axes employed will always be rectangular, unless the contrary is stated.

Examples.

Locate the points whose coördinates are as follows:
$\mathbf{x} = 3, \mathbf{y} = 2$; $\mathbf{x} = 3, \mathbf{y} = -4$; $\mathbf{x} = -2, \mathbf{y} = 5$; $\mathbf{x} = 0, \mathbf{y} = 3$; $\mathbf{x} = -2, \mathbf{y} = 0$; $\mathbf{x} = -2, \mathbf{y} = -3$, $\mathbf{x} = 0, \mathbf{y} = -2$; $\mathbf{x} = 0, \mathbf{y} = 0$, etc., etc.

74. Since the position of a point is completely determined by either the method of polar or of Cartesian coördinates, it must be possible to find an expression for the value of the coördinates of any point in one of these systems in terms of the coördinates of

the same point in the other system; i.e., to translate the expressions of one system to those belonging to the other.

75. Problem: To find expressions in polar coördinates for $\mathbf{x}$ and $\mathbf{y}$, when the pole coincides with the origin of rectangular Cartesian coördinates, and the initial line with the axis of abscissas.

In Fig. 22 we have, from § 44,

$$\cos\theta = \frac{\overline{AQ}}{\overline{AP}}, = \frac{\mathbf{x}}{\mathbf{r}}. \qquad \therefore \qquad \mathbf{x} = \mathbf{r}\cos\theta$$

and, from § 41, $$\sin\theta = \frac{\overline{QP}}{\overline{AP}} = \frac{\mathbf{y}}{\mathbf{r}}, \qquad \therefore \qquad \mathbf{y} = \mathbf{r}\sin\theta;$$

the formulæ required.

76. Problem: To find expressions in rectangular coördinates for $\mathbf{r}$ and θ, the position of the pole and initial line being as above.

In the same figure (§§ 40, 45)

$$\tan\theta = \frac{\overline{QP}}{\overline{AQ}} = \frac{\mathbf{y}}{\mathbf{x}} \qquad \text{and} \qquad \cot\theta = \frac{\overline{AQ}}{\overline{QP}} = \frac{\mathbf{x}}{\mathbf{y}}.$$

Hence, $$\theta = \tan^{-}\frac{\mathbf{y}}{\mathbf{x}} = \cot^{-}\frac{\mathbf{x}}{\mathbf{y}}.$$

Also (§ 47), $$\overline{AP}^2 = \overline{AQ}^2 + \overline{QP}^2,$$

therefore, $$\mathbf{r} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}.$$

77. We have seen that all points may be denoted by positive coördinates in the polar system, while for some points the Cartesian coördinates must be negative. Hence, if the formulæ of § 75 are to determine anything more than the *numerical* values of $\mathbf{x}$ and $\mathbf{y}$, we must consider $\sin\theta$ and $\cos\theta$ (and therefore, of course, $\tan\theta$ and $\cot\theta$) as affected with positive or negative signs, which will depend upon the magnitude of θ. And these functions are, therefore, universally assigned their algebraic signs, in accordance with the following four formulæ (deduced above), viz:

$$\sin\theta = \frac{\mathbf{y}}{\mathbf{r}},\ \cos\theta = \frac{\mathbf{x}}{\mathbf{r}},\ \tan\theta = \frac{\mathbf{y}}{\mathbf{x}},\ \text{and}\ \cot\theta = \frac{\mathbf{x}}{\mathbf{y}}.$$

In these formulæ, $\mathbf{r}$ is *always considered positive* (§ 65), which, it will be seen, is entirely consistent with the formula of § 76, $\mathbf{r} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$, since, whatever the sign of $\mathbf{x}$ or $\mathbf{y}$, their squares are positive, and the square root of the sum of these squares may be taken as positive.

SIGNS OF THE FUNCTIONS OF ANGLES.

78. The sine of a positive angle less than 180° is positive, of an angle between 180° and 360° negative.

Since the Cartesian axes here used are rectangular, the value of θ for any point in the first angle is between 0° and 90°; in the second angle, between 90° and 180°; in the third, between 180° and 270°; and in the fourth, between 270° and 360°. Now, since $\sin\theta = \frac{\mathbf{y}}{\mathbf{r}}$ and $\mathbf{r}$ is positive, $\sin\theta$ must have the same sign as $\mathbf{y}$, which is positive in the first and second angles, and negative in the third and fourth. (§ 72.)

79. The cosine of a positive angle less than 90° is positive, of an angle between 90° and 270°, negative, and of an angle between 270° and 360°, positive.

Since $\cos\theta = \frac{\mathbf{x}}{\mathbf{r}}$, $\cos\theta$ must have the same sign as $\mathbf{x}$, which is positive in the first and fourth angles, and negative in the second and third.

80. The tangent and cotangent of a positive angle less than 90° are positive, of an angle between 90° and 180°, negative, of an angle between 180° and 270, positive, and of an angle between 270° and 360°, negative.

Since $\tan\theta = \frac{\mathbf{y}}{\mathbf{x}}$ and $\cot\theta = \frac{\mathbf{x}}{\mathbf{y}}$, these functions will be positive when $\mathbf{x}$ and $\mathbf{y}$ have like signs, and negative when their signs are different. But from § 72 we see that the signs of $\mathbf{x}$ and $\mathbf{y}$ are alike in the first and third angles, and unlike in the second and fourth.

81. Any function of an angle will be in all respects equal to the same function of that angle increased or diminished by 360°, or by any multiple of that quantity by a whole number.

That is,
$$\left.\begin{matrix}\sin\\ \tan\\ \cos\\ \cot\end{matrix}\right\}\theta = \left.\begin{matrix}\sin\\ \tan\\ \cos\\ \cot\end{matrix}\right\}(\theta \pm \mathbf{n}\ 360°),$$

where **n** is an integer.

For supposing **r** to retain the same value, the point indicated is the same (§ 60), hence, the values of **r**, **x**, and **y**, will all be unchanged, numerically and algebraically.

82. Any function of an angle is *numerically* equal to the same function of that angle increased or diminished by 180°.

See Fig. 22, where it is evident that, since the divergence of the line AB′ from AX′ is the same as that of AB from AX, the values of **r**, **x**, and **y** will be numerically equal for two points situated at equal distances from A, one on each line. (§ 38.) Hence the *ratios* of these quantities will be numerically equal.

83. If two angles differ by 180°, their sines will have different signs, also their cosines; but their tangents and cotangents will have like signs.

For if the smaller angle is between 0° and 90° the larger will be between 180° and 270°; if the smaller is between 90° and 180° the larger will be between 270° and 360°; if the smaller is between 180° and 270° the larger will be between 360° and 450°, and its functions will be the same as those of an angle between 0° and 90°; if the smaller is between 270° and 360° the functions of the larger are those of an angle between 90° and 180°, etc. Now, from §§ 78–80, it will be seen that the sine of the larger angle is negative whenever that of the smaller is positive, and *vice versa*, that the same relation exists between their cosines, and that when the tangent of one is positive, the other is so also, etc., etc.

84. Combining the results of §§ 82, 83, we may write the formulæ,

$\sin(180° + a) = - \sin a,$ $\cos(180° + a) = - \cos a,$

$\tan(180° + a) = \tan a,$ $\cot(180° + a) = \cot a.$

85. Any function of a negative angle is *numerically* equal to the same function of an equal positive angle.

In Fig. 22 it is evident that the divergence of AB from AX is the same as that of AC from AX, and if two points are taken, one on each line, at equal distances from A, the value of **r**, **x**, and **y** are numerically equal for the two points, (§ 38,) whence the proposition follows.

86. The cosine of a negative angle has the same sign as the cosine of an equal positive angle; but the sine, tangent, and cotangent of a negative angle have the opposite signs from the corresponding functions of an equal positive angle.

If the positive angle be between 0° and 90°, the negative angle will have the snme functions as an angle between 270° and 360° (§ 81); if the positive be between 90° and 180°, the negative will have the functions of an angle between 180° and 270°, etc., etc.; and on applying the principles of §§ 78, 80, it will be seen in every case that the cosines of the positive and negative angles agree in sign, but that when the sine of either is negative, that of the other is positive, and *vice versa;* and that their tangents and cotangents also have opposite signs.

87. By comparison of §§ 85, 86, we may write:

$\sin(-a) = - \sin a,$ $\tan(-a) = - \tan a,$

$\cos(-a) = \cos a,$ $\cot(-a) = - \cot a.$

88. Any function of $-a$ equals the same function of $(-a + 360°)$, or $(360° - a)$, (§ 81,) hence,

$\sin(360° - a) = - \sin a;$ $\tan(360° - a) = - \tan a;$

$\cos(360° - a) = \cos a;$ and $\cot(360° - a) = - \cot a.$

89. If $\beta = 180° + a$, $\sin(180° - a) = \sin[360° - (180° + a)]$

$= \sin(360° - \beta) = -\sin\beta = -\sin(180° + \alpha) = -(-\sin\alpha) = \sin\alpha$. (See §§ 88, 83.)

Also, $\cos(180° - \alpha) = \cos[360° - (180° + \alpha)] = \cos(360 - \beta) = \cos\beta = \cos(180° + \alpha) = -\cos\alpha$.

In the same way,

$\tan(180° - \alpha) = \tan(360° - \beta) = -\tan\beta = -\tan(180° + \alpha) = -\tan\alpha$.

And $\cot(180° - \alpha) = \cot(360° - \beta) = -\cot\beta = -\cot(180° + \alpha) = -\cot\alpha$.

This result may be thus stated in words:

Any function of an angle is numerically equal to the same function of its supplement, and the sines of two angles which are supplements have like signs, but their tangents, cosines, and cotangents have unlike signs.

90. From the definition of § 42 we have,

$$\sin(90° - \alpha) = \cos\alpha,$$

and $$\tan(90° - \alpha) = \cot\alpha;$$

whence, $$\cos(90° - \alpha) = \sin\alpha,$$

and $$\cot(90° - \alpha) = \tan\alpha.$$

91. $90° + \alpha$ is the supplement of $90° - \alpha$; hence (§§ 89, 90),

$\sin(90° + \alpha) = \sin(90° - \alpha) = \cos\alpha;$

$\tan(90° + \alpha) = -\tan(90° - \alpha) = -\cot\alpha;$

$\cos(90° + \alpha) = -\cos(90° - \alpha) = -\sin\alpha$; and,

$\cot(90° + \alpha) = -\cot(90° - \alpha) = -\tan\alpha$.

92. $\sin(270° - \alpha) = \sin[90° + (180° - \alpha)] =$ (§ 91), $\cos(180° - \alpha) =$ (§ 89) $-\cos\alpha$;

$\tan(270° - \alpha) = \tan[90° + (180° - \alpha)] = -\cot(180° - \alpha) = -(-\cot\alpha) = \cot\alpha;$

$\cos(270° - \alpha) = \cos[90° + (180° - \alpha)] = -\sin(180° - \alpha) = -\sin\alpha;$

$\cot(270° - \alpha) = \cot[90° + (180° - \alpha)] = -\tan(180° - \alpha) = -(-\tan\alpha) = \tan\alpha.$

93. $\sin(270° + \alpha) = \sin[90° + (180° + \alpha)] =$ (§ 91), $\cos(180° + \alpha) =$ (§ 84), $-\cos\alpha;$

$\tan(270° + \alpha) = \tan[90° + (180° + \alpha)] = -\cot(180° + \alpha) = -\cot\alpha;$

$$\cos(270° + \alpha) = \cos[90° + (180° + \alpha)] = -\sin(180° + \alpha) = -(-\sin\alpha) = \sin\alpha;$$

$$\cot(270° + \alpha) = \cot[90° + (180° + \alpha)] = -\tan(180° + \alpha) = -\tan\alpha.$$

94. The results of §§ 78 to 93 are collected in the following tables, which the student should carefully memorize, and may extend indefinitely by the principle of § 81:

TABLE I.

	0° to 90°	90° to 180°	180° to 270°	270° to 360°
sine	+	+	—	—
tangent	+	—	+	—
cosine	+	—	—	+
cotangent	+	—	+	—

TABLE II.

α	$90°-\alpha$	$90°+\alpha$	$180°-\alpha$	$180°+\alpha$	$270°-\alpha$	$270°+\alpha$	$360°-\alpha$	$360°+\alpha$
$\sin\alpha$	$\cos\alpha$	$\cos\alpha$	$\sin\alpha$	$-\sin\alpha$	$-\cos\alpha$	$-\cos\alpha$	$-\sin\alpha$	$\sin\alpha$
$\tan\alpha$	$\cot\alpha$	$-\cot\alpha$	$-\tan\alpha$	$\tan\alpha$	$\cot\alpha$	$-\cot\alpha$	$-\tan\alpha$	$\tan\alpha$
$\cos\alpha$	$\sin\alpha$	$-\sin\alpha$	$-\cos\alpha$	$-\cos\alpha$	$-\sin\alpha$	$\sin\alpha$	$\cos\alpha$	$\cos\alpha$
$\cot\alpha$	$\tan\alpha$	$-\tan\alpha$	$-\cot\alpha$	$\cot\alpha$	$\tan\alpha$	$-\tan\alpha$	$-\cot\alpha$	$\cot\alpha$

In memorizing, the student will be aided by noticing that the signs in the first *two* columns of Table II. correspond with those in the *first* column of Table I., in the next two, to those in the second column, etc.; also, that the function *changes name* (from sine to cosine, etc.) when the angle to which $\pm\alpha$ is added is a multiple of 90° by an odd number, but does not change when that angle is a multiple of 90° by an even number.

From § 71 we see that the extension of either table in either direction will be simply a repetition, in the same order, of four columns in the case of Table I. and of eight in the case of Table II. When reference is to be made to these tables in future, they are to be considered as extended in this manner if necessary.

95. From Table I. it appears

(*a.*) That like functions of two angles must have the same sign unless some multiple of 90°, (including 0°, which $= 0 \times 90°$,) is intermediate in value between the angles. Also,

(*b.*) From Table II., that a sine numerically equal to $\sin \alpha$ belongs to one angle between 0° and 90°, to one between 90° and 180°, etc., also to one between 0° and — 90°, etc.: and that a sine numerically equal to $\cos \alpha$ likewise belongs to an angle between 0° and 90°, to one between 90° and 180°, etc.; and that tangents, cosines, and cotangents are similarly distributed; whence

(*c.*) It follows that any function of any positive or negative angle whatever is equal to the *like* function of some angle between the limits of 0° and 90°, inclusive; which angle may be found by adding or subtracting the requisite multiple of 90°.

96. Table II. is *complete*, i. e., it embraces *all* the angles whose sine is numerically equal to $\sin \alpha$ or to $\cos \alpha$, whose tangent is numerically equal to $\tan \alpha$ or to $\cot \alpha$, etc. For, if it be incomplete, there must be at least *two* angles [see § 95 (*b*)] such that no multiple of 90° is intermediate between them (that is, *two* between 0° and 90°, or *two* between 90° and 180°, etc.,) having like functions which are numerically equal. If so, those functions would also be algebraically identical, [§ 95 (*a*,)] and, by adding or subtracting the requisite multiple of 90°, two angles between 0° and 90° might be found, whose like functions would also be be equal. [§ 95 (*c.*)] Hence, if it can be shown to be impossible that *two* angles between 0° and + 90° should have like functions which are equal, Table II. is complete. Now, let θ and θ' be two positive angles, each less than 90°, made with AX (Fig. 23) by the two lines AB and AC, and let us suppose that the *sines* (e. g.) of these angles are equal. Let P be any point on AB, and through P let a line pass parallel to the axis of abscissas AX, and cutting the axis of ordinates at the point R, and the line AC at Q. Let $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{r}$ and θ be the coördinates of P, and $\mathbf{x}'$ and $\mathbf{y}'$, $\mathbf{r}'$ and θ' those of Q. Then, by construction, $\mathbf{y} = \mathbf{y}'$;
and, by hypothesis,

$$\sin \theta = \sin \theta' \text{ or } \frac{\mathbf{y}}{\mathbf{r}} = \frac{\mathbf{y}'}{\mathbf{r}'} \quad \therefore \quad \mathbf{r} = \mathbf{r}'.$$

Now, since P and Q are both in the first angle, $\mathbf{x}$ and $\mathbf{x}'$ have like signs, but

$$\mathbf{x} = \sqrt{\mathbf{r}^2 - \mathbf{y}^2} \quad \text{and} \quad \mathbf{x}' = \sqrt{\mathbf{r}'^2 - \mathbf{y}'^2} \quad \therefore \quad \mathbf{x} = \mathbf{x}'.$$

Hence P and Q coincide; and thus every point in AB coincides with a point in AC, hence $\theta = \theta'$. In like manner it may be proved that $\theta = \theta'$ if $\cos\theta = \cos\theta'$, etc. Hence the proposition is proved.

FUNCTIONS OF THE SUM AND DIFFERENCE OF ANGLES AND OF MULTIPLE ANGLES.

97. The sine of the sum of two angles is equal to the product of the sine of the first by the cosine of the second, *plus* the product of the cosine of the first by the sine of the second; i. e.,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

The sine of the difference of two angles is equal to the product of the sine of the first by the cosine of the second, *minus* the product of the cosine of the first by the sine of the second; i. e.,

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

The cosine of the sum of two angles is equal to the product of their cosines, *minus* the product of their sines; i. e.,

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

The cosine of the difference of two angles is equal to the product of their cosines, *plus* the product of their sines; i. e.,

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

These propositions will first be proved on the supposition that each of the angles is numerically less than 90°; and the demonstration will then be extended to embrace all angles.

The values of the functions of $(-\alpha - \beta)$ and $(-\alpha + \beta)$ may be determined by § 87 when we have found the functions of $(\alpha + \beta)$ and $(\alpha - \beta)$; hence we need examine only the cases in which the larger angle α is positive.

In Fig. 24, 25, or 26, let the line AP make the positive angle α

with the line AX, and let the line AQ make the angle β with the line AP — positive in Fig. 24 or 25, but negative in Fig. 26 — so that in the former case the angle made by AQ with AX is equal to $\alpha + \beta$, but in the latter case to $\alpha - \beta$. Through P, any point on AP, let a line, PR, pass perpendicular to AX, and meeting it at R, and another, PN, parallel to AX; also a line, PQ, perpendicular to AP, and meeting AQ at Q. Through Q let a line, QS, pass, parallel to PR, and therefore (§ 26) perpendicular to AX, and meeting AX at S, AP at T, and PN at N. Then the angle formed by PN and PT is equal to α, the angle formed by AP with AX (§ 26). But this angle is the complement of that formed by PN and PQ (which therefore equals $90° - \alpha$), and this, again, is (§ 30) the complement of the angle formed by QP and QN. The latter angle, therefore, is equal to α. Hence (§§ 41, 44),

$$\sin\alpha = \frac{\overline{RP}}{\overline{AP}} \text{ or } \frac{\overline{NP}}{\overline{QP}}, \qquad \sin\beta = \frac{\overline{PQ}}{\overline{AQ}},$$

$$\cos\alpha = \frac{\overline{AR}}{\overline{AP}} \text{ or } \frac{\overline{QN}}{\overline{QP}}, \quad \text{and} \quad \cos\beta = \frac{\overline{AP}}{\overline{AQ}}.$$

Also the distance $\overline{RP} = \overline{SN}$ (§ 35).

Now, in Fig. 24 or 25, $\sin(\alpha + \beta) = \frac{\overline{SQ}}{\overline{AQ}}$;

but $\overline{SQ} = \overline{SN} + \overline{NQ} = \overline{RP} + \overline{NQ}$.

Whence
$$\frac{\overline{SQ}}{\overline{AQ}} = \frac{\overline{RP}}{\overline{AQ}} + \frac{\overline{NQ}}{\overline{AQ}}.$$

Now,
$$\frac{\overline{RP}}{\overline{AQ}} = \frac{\overline{RP}}{\overline{AP}} \times \frac{\overline{AP}}{\overline{AQ}} = \sin\alpha\cos\beta;$$

and
$$\frac{\overline{NQ}}{\overline{AQ}} = \frac{\overline{QN}}{\overline{QP}} \times \frac{\overline{PQ}}{\overline{AQ}} = \cos\alpha\sin\beta.$$

Substituting these values it appears that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

In like manner, in Fig. 26,

$$\sin(\alpha - \beta) = \frac{\overline{SQ}}{\overline{AQ}} = \frac{\overline{RP}}{\overline{AQ}} - \frac{\overline{NQ}}{\overline{AQ}} =$$

$$\frac{\overline{RP}}{\overline{AP}} \times \frac{\overline{AP}}{\overline{AQ}} - \frac{\overline{QN}}{\overline{QP}} \times \frac{\overline{PQ}}{\overline{AQ}} = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

[It should be noticed that this formula may be obtained directly from the other; for, by substituting $(-\beta)$ for β in

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

since $\sin(-\beta) = -\sin\beta$, and $\cos(-\beta) = \cos\beta$,

we have $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.]

98. $\text{Cos}(\alpha+\beta) = \sin[90° - (\alpha+\beta)]$ (§ 86),

$$= \sin[(90° - \alpha) - \beta] = \sin(90° - \alpha)\cos\beta - \cos(90° - \alpha)\sin\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

$$\text{Cos}(\alpha-\beta) = \sin[90° - (\alpha-\beta)] = \sin[(90° - \alpha) + \beta] = \sin(90° - \alpha)\cos\beta + \cos(90° - \alpha)\sin\beta = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

99. These formulæ have now been demonstrated for the case in which α and β are each less than 90°. If either or each of them is numerically greater than 90°, let γ and δ be two angles each less than 90°, which differ from α and β respectively by some multiple of 90° by a whole number. Then the difference between $\alpha+\beta$ and $\gamma+\delta$ will be either zero or a multiple of 90°, and the same will be true of the difference between $\alpha-\beta$ and $\gamma-\delta$. Now, the demonstrations of the two preceding §§ furnish expressions for the sine and cosine of $\gamma+\delta$ and $\gamma-\delta$; and from these, by Table II. (§ 94), we may find the like functions of $\alpha+\beta$ and $\alpha-\beta$. In every case we shall find that

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta,\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta,\\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta,\\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.
\end{aligned}$$

Hence these formulæ are true for *all* values of α and β.

For instance, let α be an angle between $-90°$ and $-180°$, and let β be an angle between $+270°$ and $+360°$. Then $\alpha = \gamma - 180°$ and $\beta = \delta + 270°$, where both γ and δ are between 0° and 90°. Hence,

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin[(\gamma - 180°) + (\delta + 270°)]\\
&= \sin[(\gamma+\delta) + 80°] = \cos(\gamma+\delta)\\
&= \cos\gamma\cos\delta - \sin\gamma\sin\delta.
\end{aligned}$$

Restoring α and β to the equation, by substituting $(180 + \alpha)$ for γ and $(\beta - 270°)$ for δ, this expression becomes

$$\cos(180° + \alpha)\cos(\beta - 270°) - \sin(180° + \alpha)\sin(\beta - 270°)$$
$$= \text{(see § 87)} \cos(180° + \alpha)\cos(270° - \beta) + \sin(180° + \alpha)\sin(270° - \beta)$$
$$= \text{(by Table II.)} (-\cos\alpha)(-\sin\beta) + (-\sin\alpha)(-\cos\beta)$$
$$= \cos\alpha\sin\beta + \sin\alpha\cos\beta$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

In the same way, we may find

$$\cos(\alpha - \beta) = \cos(\gamma - \delta - 450°) = \sin(\gamma - \delta)$$
$$= \sin\gamma\cos\delta - \cos\gamma\sin\delta$$
$$= (-\sin\alpha)(-\sin\beta) - (-\cos\alpha\cos\beta)$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Similar processes will lead to similar results for any other values of α and β.

100. The tangent of the sum of two angles is equal to the sum of their tangents, divided by 1 *minus* the product of their tangents; i.e.,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

The tangent of the difference of two angles is equal to the difference of their tangents, divided by 1 *plus* the product of their tangents; i.e.,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

The cotangent of the sum of two angles is equal to the product of their cotangents *minus* 1, divided by the sum of their cotangents; i.e.,

$$\cot(\alpha + \beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}.$$

The cotangent of the difference of two angles is equal to the product of their cotangents *plus* 1, divided by the difference of their cotangents taken in the contrary order; i.e.,

$$\cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}.$$

These formulæ are readily deduced from those of § 97 by the use of the relations $\tan = \dfrac{\sin}{\cos}$ and $\cot = \dfrac{\cos}{\sin}$. Thus:—

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Dividing both terms of this fraction by $\cos\alpha\cos\beta$, we have

$$\tan(\alpha + \beta) = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha} \times \dfrac{\sin\beta}{\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

In like manner,

$$\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}}{1 + \dfrac{\sin\alpha}{\cos\alpha} \times \dfrac{\sin\beta}{\cos\beta}} = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

In deducing the formulæ for cotangents, both terms of the fraction are divided by $\sin\alpha\sin\beta$, thus:—

$$\cot(\alpha + \beta) = \frac{\cos(\alpha + \beta)}{\sin(\alpha + \beta)} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

$$= \frac{\dfrac{\cos\alpha}{\sin\alpha} \times \dfrac{\cos\beta}{\sin\beta} - 1}{\dfrac{\cos\beta}{\sin\beta} + \dfrac{\cos\alpha}{\sin\alpha}} = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}.$$

$$\cot(\alpha - \beta) = \frac{\cos(\alpha - \beta)}{\sin(\alpha - \beta)} = \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

$$= \frac{\dfrac{\cos\alpha}{\sin\alpha} \times \dfrac{\cos\beta}{\sin\beta} + 1}{\dfrac{\cos\beta}{\sin\beta} - \dfrac{\cos\alpha}{\sin\alpha}} = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}.$$

101. The formulæ of §§ 97–100 are collected in the following table:

TABLE III.

A. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$

B. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

C. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

D. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

E. $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$

F. $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$

G. $\cot(\alpha + \beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}.$

H. $\cot(\alpha - \beta) = \dfrac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}.$

102. If, in formulæ A, C, E, and G of Table III., we suppose $\beta = \alpha$, we may write at once the following table of functions of *double* angles: —

$$\sin 2\alpha = 2\sin\alpha\cos\alpha.$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}.$$
$$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}.$$

Since $\cos^2\alpha = 1 - \sin^2\alpha$, and $\sin^2\alpha = 1 - \cos^2\alpha$, (§ 48,) we may write the value of $\cos 2\alpha$ in either of the following forms: —

$$\cos 2\alpha = 2\cos^2\alpha - 1, \quad \text{or} \quad \cos 2\alpha = 1 - 2\sin^2\alpha.$$

Also, since $\dfrac{1}{\cot\alpha} = \tan\alpha,$

$$\cot 2\alpha = \tfrac{1}{2}(\cot\alpha - \tan\alpha).$$

103. $\sin 3\alpha = \sin(2\alpha + \alpha) = \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha = 2\sin\alpha\cos^2\alpha + (\cos^2\alpha - \sin^2\alpha)\sin^2\alpha = 3\sin\alpha\cos^2\alpha - \sin^3\alpha.$

If in this value we substitute for $\cos^2\alpha$, $1 - \sin^2\alpha$, we have

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha.$$

In like manner we may find that

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha,$$
$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha},$$
$$\cot 3\alpha = \frac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1}.$$

$$\text{Sin } 4\,a = \sin\,(2 \times 2)\,a = 2 \sin 2\,a \cos 2\,a$$
$$= 4 \sin a \cos a\,(\cos^2 a - \sin^2 a).$$

In like manner we may find the other functions of $4\,a$. The functions of $5\,a$ may be found by regarding it as the sum of $4\,a$ and a, or of $3\,a$, and $2\,a$, etc.

104. Since $\quad \cos 2\,a = 1 - 2 \sin^2 a,$

$$\sin^2 a = \tfrac{1}{2}\,(1 - \cos 2\,a);$$

and since $\quad \cos 2\,a = 2 \cos^2 a - 1,$

$$\cos^2 a = \tfrac{1}{2}\,(1 + \cos 2\,a).$$

Since no limitation has been put on the values of a and $2\,a$, save that one is double the other, we may write the above equations

$$\sin^2 \tfrac{1}{2}\,a = \tfrac{1}{2}\,(1 - \cos a) \quad \text{and} \quad \cos^2 \tfrac{1}{2}\,a = \tfrac{1}{2}\,(1 + \cos a),$$

whence

$$\sin \tfrac{1}{2}\,a = \sqrt{\tfrac{1}{2}\,(1 - \cos a)} \quad \text{and} \quad \cos \tfrac{1}{2}\,a = \sqrt{\tfrac{1}{2}\,(1 + \cos a)}.$$

Also,

$$\tan \tfrac{1}{2}\,a = \frac{\sin \frac{1}{2}\,a}{\cos \frac{1}{2}\,a} = \sqrt{\frac{1 - \cos a}{1 + \cos a}},$$

and

$$\cot \tfrac{1}{2}\,a = \frac{\cos \frac{1}{2}\,a}{\sin \frac{1}{2}\,a} = \sqrt{\frac{1 + \cos a}{1 - \cos a}}.$$

Other forms of the values of $\tan \frac{1}{2}\,a$ and $\cot \frac{1}{2}\,a$ may be obtained from the values of $\tan 2\,a$ and $\cot 2\,a$ given in § 102. The signs of all these functions of half angles are determinate in every case, being fixed by the magnitude of the angles.

PARTICULAR VALUES OF THE FUNCTIONS.

105. If, in formulæ B, D, F, and H, of § 101, we make $\beta = a$, we shall have $\sin\,(a - a) = \sin 0 = \sin a \cos a - \sin a \cos a = 0$.

$$\cos 0° = \cos^2 a + \sin^2 a = 1, \text{ (§ 48,)}$$

$$\tan 0° = \frac{0}{1 + \tan^2 a} = 0,$$

$$\cot 0° = \frac{\cot^2 a + 1}{0} = \infty.$$

By reference to Table II. (§ 94), we may now find the values of the functions of 90°, 180°, etc. (since 90° = 90°

$\pm$ 0°, etc.) These values are collected in the following table, the *signs* in which are taken from Table I., and indicate the signs of angles a little less and a little greater than the given angle. Thus, the sign of 180° is written ± 0; by which is meant the sin 180° $= 0$, and that an angle a little less than 180° has a positive sine, while an angle a little greater has a negative sine. (In the ambiguous sign $\pm$ or $\mp$, the upper sign belongs to the function of the angle a little less than the given angle, and the lower sign to that of the angle a little greater.) When only one sign is written, it is understood that the function has that sign, both for the given value of the angle, and for values a little greater and a little less.

TABLE IV.

	0°	90°	180°	270°	360°
Sine	∓ 0	$+ 1$	± 0	$- 1$	∓ 0
tangent	∓ 0	$\pm \infty$	∓ 0	$\pm \infty$	∓ 0
cosine	$+ 1$	± 0	$- 1$	∓ 0	$+ 1$
cotangent	$\mp \infty$	± 0	$\mp \infty$	± 0	$\mp \infty$

This table, like Tables I. and II. may be extended indefinitely in either direction, and such an extension will be simply a repetition of *four* different columns.

106. From § 96, we know that the angles included in table IV. are the *only* angles having the given functions; i.e., that no *other* angles can have sines or tangents equal to 0, except 0°, 180°, 360°, and these increased or diminished by the multiples of 180°; no cosine can be equal to $+ 1$ except those of 0° and multiples of 360°, positive or negative, etc. Moreover, the values given in this table are the *limiting values* of the function; that is, they are, algebraically or numerically, the greatest and least values possible for the functions. It is evident that no finite quantity can be greater than ∞, or numerically less than 0, or algebraically less than $- \infty$, and these values embrace all given in the table, except those of the sines of 90°, 270°, etc., and the cosines of 0°, 180°, etc., which are equal to $+ 1$ or $- 1$. Hence it is to be shown only that no sine or cosine can be numerically greater than 1. This readily appears from the equation $\sin^2 a + \cos^2 a = 1$ (§ 48), where $\sin^2 a$ and $\cos^2 a$ must be positive, whether $\sin a$ and $\cos a$

are so or not, therefore neither of them can exceed 1; and hence neither sin a or cos a can numerically exceed 1, whatever the value of a.

107. Hence the sines of angles between 0° and 90° are intermediate in value between 0 and $+1$, their cosines are between $+1$ and 0, their tangents between 0 and $+\infty$, etc. The sines of angles between 90° and 180° are intermediate in value between $+1$ and 0, etc. If θ be an angle supposed to increase continuously in value, commencing at 0°. sin θ will be 0 when θ is 0, and will increase gradually till $\theta = 90°$, when it will be $+1$, then it will diminish until $\theta = 180°$, when it will again equal 0; as θ passes from 180° to 270°, sin θ will become negative and will continue to diminish algebraically, but increase numerically, till, when $\theta = 270°$, sin $\theta = -1$. From that point it will increase algebraically, but decrease numerically, till $\theta = 360°$, when sin θ again equals 0, and thence the same changes are repeated in the same order.

Tan θ, when $\theta = 0$, is also equal to 0, and, like sin θ, increases as θ approaches 90°, but more rapidly, so that when $\theta = 90°$, tan $\theta = \infty$. Tan θ has been positive up to this point, but here changes sign from $+\infty$ to $-\infty$, and as θ increases in value toward 180°, continues to decrease numerically, but increase algebraically to 0, which value it reaches when $\theta = 180°$. In passing through this value it changes sign, and continues to increase till $\theta = 270°$, when it is again infinite, and again changes sign, and from this point decreases numerically till it becomes equal to 0° when $\theta = 360°$.

[Let the student trace the corresponding changes in the value of cos θ and cot θ, as well as the values of all the functions when θ is a *negative* angle, increasing numerically; and let him notice that the functions of angles change their sign when, and *only* when, these functions pass through the values of 0 and ∞. It is an established principle in Geometry as well as Algebra, that *no* varying function can change sign, except in passing through one of these values. The converse proposition, however, that a function in passing through 0 or ∞ *always* changes sign, though true of these four functions of angles, as we have seen, is *not* a principle of general application.]

108. Besides the angles which are multiples of 90°, and whose functions have the *limiting* values, there are other angles whose functions are of frequent use; particularly 45°, 30°, and 60°.

If $a = 45°$, $a = 90° - a$, $\therefore$ $\sin a = \sin (90° - a) = \cos a$.

But $\sin^2 a + \cos^2 a = 1$ (§ 48) $\therefore$ $2 \sin^2 a = 1$,

or $\sin a = \cos a = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2} \sqrt{2}$.

Now, as $\tan a = \dfrac{\sin a}{\cos a}$ and $\cot a = \dfrac{\cos a}{\sin a}$,

when $\sin a = \cos a$, each of these is equal to 1.

$$\therefore \quad \sin 45° = \tfrac{1}{2} \sqrt{2}, \qquad \cos 45° = \tfrac{1}{2} \sqrt{2},$$
$$\tan 45° = 1, \qquad \cot 45° = 1.$$

If $a = 30°$, $90° - a = 2a$, or $\sin (90° - a) = \sin 2a$,

$$\therefore \quad \cos a = 2 \sin a \cos a; \qquad \text{whence, } \sin a = \tfrac{1}{2};$$
$$\cos a = \sqrt{1 - \sin^2 a} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2} \sqrt{3};$$
$$\cot a = \frac{\cos a}{\sin a} = \sqrt{3};$$
$$\tan a = \frac{1}{\cot a} = \sqrt{\tfrac{1}{3}} = \tfrac{1}{3} \sqrt{3}.$$
$$\therefore \quad \sin 30° = \tfrac{1}{2}; \qquad \cos 30° = \tfrac{1}{2} \sqrt{3};$$
$$\tan 30° = \tfrac{1}{3} \sqrt{3}; \qquad \cot 30° = \sqrt{3}.$$

From the cosine of 30° we may find the sine and cosine of 15° by the formulæ of § 104, and from these the tangent and cotangent. To find the functions of 60°, we may employ the formulæ of § 102; or, more simply, since 60° is the complement of 30°, we may write $\sin 60° = \cos 30°$, $\cos 60° = \sin 30°$, etc.

$$\therefore \quad \sin 60° = \tfrac{1}{2} \sqrt{3}; \qquad \cos 60° = \tfrac{1}{2};$$
$$\tan 60° = \sqrt{3}; \qquad \cot 60° = \tfrac{1}{3} \sqrt{3}.$$

109. In a similar manner we may find the functions of 18°.

If $a = 18°$, we have $2a = 90° - 3a$.

Hence (§§ 102, 103),

$$2 \sin a \cos a = 4 \cos^3 a - 3 \cos a,$$

or

$$2 \sin a = 4 \cos^2 a - 3.$$

Substituting for $\cos^2 a$ its value, $1 - \sin^2 a$, this becomes

$$2 \sin a = 4 - 4 \sin^2 a - 3,$$

or

$$4 \sin^2 a + 2 \sin a = 1,$$

whence, by the solution of a quadratic equation, we have

$$\sin a = \tfrac{1}{4} (\sqrt{5} - 1).$$

Only one of the roots of this equation (the positive one) can be taken as the sine of 18°, since the angle is less than 90°. For a similar reason, all the functions in the preceding section are positive.

110. *To compute the sine and cosine of* 1′.—From the formulæ of §§ 102 and 103 we may obtain the equations:

$$\frac{\sin 2a}{2 \sin a} = \cos a = \sqrt{1 - \sin^2 a},$$

$$\frac{\sin 3a}{3 \sin a} = 1 - \tfrac{4}{3} \sin^2 a,$$

$$\frac{\sin 4a}{4 \sin a} = (1 - 2 \sin^2 a) \sqrt{1 - \sin^2 a}, \qquad \text{etc.}$$

In all these values it is to be noticed that when a is a very small angle, and, consequently, $\sin a$ is a very small fraction, $\sin^2 a$ will be much smaller, and the second member of each equation will be very nearly equal to 1. Hence, when a is a very small angle,

$$\frac{\sin 2a}{\sin a} = 2, \quad \frac{\sin 3a}{\sin a} = 3, \quad \frac{\sin 4a}{\sin a} = 4, \quad \text{etc., } \textit{very nearly;}$$

that is, *the sines of small angles are very nearly in the same ratio as the angles themselves.* Hence, if we can find the sine of some angle not larger than 1′, we may thence find the sine of 1′, with sufficient exactness for all practical purposes, by multiplying that sine by the ratio of 1′ to the corresponding angle. In order to find the sine of this small angle, we may first calculate the cosine of 3° from the formula $\cos(a - \beta) = \cos a \cos \beta + \sin a \sin \beta$, since 3° is the difference of 18° and 15°, whose functions are known. Then, by successive applications of the formula of § 104, the cosine of the half, the quarter, etc., of 3° may be obtained, to as small an angle as we wish, and from the cosine of this angle we may find the sine of an angle half as large. In this way, having computed from the formula of § 108 the values

$$\sin 15° = .25881904510252,$$
$$\cos 15° = .96592582627196,$$

and from § 109,

$$\sin 18° = .30901699437495,$$
$$\cos 18° = .95105651629515,$$

we obtain $\cos 3° = .998629534738030;$

and thence, successively,

$$\cos 1° 30' = .99965732497149,$$
$$\cos 45' = .99991432757299,$$
$$\cos \tfrac{45}{2}' = .99997858166387,$$
$$\cos \tfrac{45}{4}' = .99999464540163,$$
$$\cos \tfrac{45}{8}' = .99999866134951,$$
$$\cos \tfrac{45}{16}' = .99999966533732,$$

and $\cos \tfrac{45}{32}' = .99999991633433.$

From $\cos \frac{45}{16}'$ we obtain

$$\sin \tfrac{45}{32}' = .000409061534,$$

and from $\cos \frac{45}{32}'$, $\sin \frac{45}{64}' = .000204530772.$

We now observe that the half of $\sin \frac{45}{32}'$ differs from $\sin \frac{45}{64}'$ by not more than 5 in the twelfth place of decimals. Hence we infer that if we take $\frac{64}{45}$ of $\sin \frac{45}{64}'$, it will be the value of the sine of $\frac{64}{45}$ of $\frac{45}{64}'$, that is, of 1', correct to at least ten or eleven places of decimals. The value to ten places is,

$$\sin 1' = .0002908882.$$

The cosine of 1' may now be found from the formula $\cos 1' = \sqrt{1 - \sin^2 1'}$. Its value is, $\cos 1' = .9999999577.$

111. *To compute a table of sines, tangents, etc.*—By adding and subtracting the first four formulæ of § 101 we obtain the following:

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin\alpha \cos\beta, \tag{1}$$
$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos\alpha \sin\beta, \tag{2}$$
$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos\alpha \cos\beta, \tag{3}$$
$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin\alpha \sin\beta. \tag{4}$$

If in (1) and (3) we make β constantly equal to 1', and let α equal successively 1', 2', 3', etc., we shall obtain, by transposition,

$$\sin(1' + 1') = \left.\begin{array}{l}\sin 2' = 2 \sin 1' \cos 1' - \sin 0' \\ \cos 2' = 2 \cos 1' \cos 1' - \cos 0'\end{array}\right\} \begin{array}{l}\text{when } \alpha = 1' \\ \text{and } \beta = 1'.\end{array}$$

$$\left.\begin{array}{l}\sin 3' = 2 \sin 2' \cos 1' - \sin 1' \\ \cos 3' = 2 \cos 2' \cos 1' - \cos 1'\end{array}\right\} \begin{array}{l}\text{when } \alpha = 2' \\ \text{and } \beta = 1'.\end{array}$$

etc., etc., etc., etc.,

and by these formulæ we may calculate successively the sine and cosine of 2', 3', 4', etc., up to 30°. We may now avoid the labor

of further multiplications by the use of the formulæ (1) and (4) of this section, in which we make α constantly equal to 30°, and let β equal 1′, 2′, 3′, etc., successively.

Remembering that $\sin 30° = \frac{1}{2}$, we have,

$$\left.\begin{array}{l}\sin 30°\ 1' = \cos 1' - \sin 29°\ 59' \\ \cos 30°\ 1' = \cos 29°\ 59' - \sin 1'\end{array}\right\} \text{when } \alpha = 30° \text{ and } \beta = 1'.$$

$$\left.\begin{array}{l}\sin 30°\ 2' = \cos 2' - \sin 29°\ 58' \\ \cos 30°\ 2' = \cos 29°\ 58' - \sin 2'\end{array}\right\} \text{when } \alpha = 30° \text{ and } \beta = 2'.$$

etc., etc., etc.

We may thus obtain by subtraction the sines and cosines of angles up to 45°. From this point, by the use of the formulæ $\sin \alpha = \cos (90° = \alpha)$ and $\cos \alpha = \sin (90° - \alpha)$, we may write at once the sines and cosines of angles from 45° to 90°. It will not be necessary to extend the table beyond 90°, since by Table II., § 94, we may find the functions of any angles whatever, from those of angles between 0° and 90°.

A table of tangents may now be formed by dividing each sine by its corresponding cosine, since $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$; and by taking each tangent thus found as the cotangent of the complementary angle, a table of cotangents is obtained.

Tables of Natural Sines and Tangents are given on pages 45 and 46, and Table of Reciprocals on pages 47 and 48.

Table V. Natural Sines.

	0′	10′	20	30′	40	50′	D		0′	10′	20′	30′	40	50	D
0°	0000	0029	0058	0087	0116	0145	2.9	45°	7071	7092	7112	7132	7153	7173	2.0
1	0175	0204	0233	0262	0291	0320	2.9	46	7193	7214	7234	7254	7274	7294	2.0
2	0349	0378	0407	0436	0465	0494	2.9	47	7314	7333	7353	7373	7392	7412	2.0
3	0523	0552	0581	0610	0640	0669	2.9	48	7431	7451	7470	7490	7509	7528	1.9
4	0698	0727	0756	0785	0814	0843	2.9	49	7547	7566	7585	7604	7623	7642	1.9
5	0872	0901	0929	0958	0987	1016	2.9	50	7660	7679	7698	7716	7735	7753	1.9
6	1045	1074	1103	1132	1161	1190	2.9	51	7771	7790	7808	7826	7844	7862	1.8
7	1219	1248	1276	1305	1334	1363	2.9	52	7880	7898	7916	7934	7951	7969	1.8
8	1392	1421	1449	1478	1507	1536	2.9	53	7986	8004	8021	8039	8056	8073	1.7
9	1564	1593	1622	1650	1679	1708	2.9	54	8090	8107	8124	8141	8158	8175	1.7
10	1736	1765	1794	1822	1851	1880	2.9	55	8192	8208	8225	8241	8258	8274	1.6
11	1908	1937	1965	1994	2022	2051	2.9	56	8290	8307	8323	8339	8355	8371	1.6
12	2079	2108	2136	2164	2193	2221	2.8	57	8387	8403	8418	8434	8450	8465	1.6
13	2250	2278	2306	2334	2363	2391	2.8	58	8480	8496	8511	8526	8542	8557	1.5
14	2419	2447	2476	2504	2532	2560	2.8	59	8572	8587	8601	8616	8631	8646	1.5
15	2588	2616	2644	2672	2700	2728	2.8	60	8660	8675	8689	8704	8718	8732	1.4
16	2756	2784	2812	2840	2868	2896	2.8	61	8746	8760	8774	8788	8802	8816	1.4
17	2924	2952	2979	3007	3035	3062	2.8	62	8829	8843	8857	8870	8883	8897	1.3
18	3090	3118	3145	3173	3201	3228	2.8	63	8910	8923	8936	8949	8962	8975	1.3
19	3256	3283	3311	3338	3365	3393	2.7	64	8988	9001	9013	9026	9038	9051	1.3
20	3420	3448	3475	3502	3529	3557	2.7	65	9063	9075	9088	9100	9112	9124	1.2
21	3584	3611	3638	3665	3692	3719	2.7	66	9135	9147	9159	9171	9182	9194	1.2
22	3746	3773	3800	3827	3854	3881	2.7	67	9205	9216	9228	9239	9250	9261	1.1
23	3907	3934	3961	3987	4014	4041	2.7	68	9272	9283	9293	9304	9315	9325	1.1
24	4067	4094	4120	4147	4173	4200	2.6	69	9336	9346	9356	9367	9377	9387	1.0
25	4226	4253	4279	4305	4331	4358	2.6	70	9397	9407	9417	9426	9436	9446	1.0
26	4384	4410	4436	4462	4488	4514	2.6	71	9455	9465	9474	9483	9492	9502	0.9
27	4540	4566	4592	4617	4643	4669	2.6	72	9511	9520	9528	9537	9646	9554	0.9
28	4695	4720	4746	4772	4797	4823	2.6	73	9563	9572	9580	9588	9596	9605	0.8
29	4848	4874	4899	4924	4950	4975	2.5	74	9613	9621	9628	9636	9644	9652	0.8
30	5000	5025	5050	5075	5100	5125	2.5	75	9659	9667	9674	9681	9689	9696	0.7
31	5150	5175	5200	5225	5250	5275	2.5	76	9703	9710	9717	9724	9730	9737	0.7
32	5299	5324	5348	5373	5398	5422	2.5	77	9744	9750	9757	9763	9769	9775	0.6
33	5446	5471	5495	5519	5544	5568	2.4	78	9781	9787	9793	9799	9805	9811	0.6
34	5592	5616	5640	5664	5688	5712	2.4	79	9816	9822	9827	9833	9838	9843	0.5
35	5736	5760	5783	5807	5831	5854	2.4	80	9848	9853	9858	9863	9868	9872	0.5
36	5878	5901	5925	5948	5972	5995	2.3	81	9877	9881	9886	9890	9894	9899	0.4
37	6018	6041	6065	6088	6111	6134	2.3	82	9903	9907	9911	9914	9918	9922	0.4
38	6157	6180	6202	6225	6248	6271	2.3	83	9925	9929	9932	9936	9939	9942	0.3
39	6293	6316	6338	6361	6383	6406	2.2	84	9945	9948	9951	9954	9957	9959	0.3
40	6428	6450	6472	6494	6517	6539	2.2	85	9962	9964	9967	9969	9971	9974	0.2
41	6561	6583	6604	6626	6648	6670	2.2	86	9976	9978	9980	9981	9983	9985	0.2
42	6691	6713	6734	6756	6777	6799	2.1	87	9986	9988	9989	9990	9992	9993	0.1
43	6820	6841	6862	6884	6905	6926	2.1	88	9994	9995	9996	9997	9997	9998	0.1
44	6947	6967	6988	7009	7030	7050	2.1	89	9998	9999	9999	unity	unity	uuity	0.0
	0′	10	20′	30′	40′	50′			0′	10′	20′	30′	40	50′	

Table VI. Natural Tangents.

	0′	10′	20′	30′	40′	50′	D		0′	10′	20′	30	40′	50′	D
0°	0000	0029	0058	0087	0116	0145	2.9	45°	1.000	1.006	1.012	1.018	1.024	1.030	0.6
1	0175	0204	0233	0262	0291	0320	2.9	46	1.036	1.042	1.048	1.054	1.060	1.066	0.6
2	0349	0378	0407	0437	0466	0495	2.9	47	1.072	1.079	1.085	1.091	1.098	1.104	0.6
3	0524	0553	0582	0612	0641	0670	2.9	48	1.111	1.117	1.124	1.130	1.137	1.144	0.7
4	0699	0729	0758	0787	0816	0846	2.9	49	1.150	1.157	1.164	1.171	1.178	1.185	0.7
5	0875	0904	0934	0963	0992	1022	2.9	50	1.192	1.199	1.206	1.213	1.220	1.228	0.7
6	1051	1080	1110	1139	1169	1198	2.9	51	1.235	1.242	1.250	1.257	1.265	1.272	0.8
7	1228	1257	1287	1317	1346	1376	3.0	52	1.280	1.288	1.295	1.303	1.311	1.319	0.8
8	1405	1435	1465	1495	1524	1554	3.0	53	1.327	1.335	1.343	1.351	1.360	1.368	0.8
9	1584	1614	1644	1673	1703	1733	3.0	54	1.376	1.385	1.393	1.402	1.411	1.419	0.9
10	1763	1793	1823	1853	1883	1914	3.0	55	1.428	1.437	1.446	1.455	1.464	1.473	0.9
11	1944	1974	2004	2035	2065	2095	3.0	56	1.483	1.492	1.501	1.511	1.520	1.530	1.0
12	2126	2156	2186	2217	2247	2278	3.1	57	1.540	1.550	1.560	1.570	1.580	1.590	1.0
13	2309	2339	2370	2401	2432	2462	3.1	58	1.600	1.611	1.621	1.632	1.643	1.653	1.1
14	2493	2524	2555	2586	2617	2648	3.1	59	1.664	1.675	1.686	1.698	1.709	1.720	1.1
15	2679	2711	2742	2773	2805	2836	3.1	60	1.732	1.744	1.756	1.767	1.780	1.792	1.2
16	2867	2899	2931	2962	2994	3026	3.2	61	1.804	1.816	1.829	1.842	1.855	1.868	1.3
17	3057	3089	3121	3153	3185	3217	3.2	62	1.881	1.894	1.907	1.921	1.935	1.949	1.4
18	3249	3281	3314	3346	3378	3411	3.2	63	1.963	1.977	1.991	2.006	2.020	2.035	1.5
19	3443	3476	3508	3541	3574	3607	3.3	64	2.050	2.066	2.081	2.097	2.112	2.128	1.6
20	3640	3673	3706	3739	3772	3805	3.3	65	2.145	2.161	2.177	2.194	2.211	2.229	1.7
21	3839	3872	3906	3939	3973	4006	3.4	66	2.246	2.264	2.282	2.300	2.318	2.337	1.8
22	4040	4074	4108	4142	4176	42 0	3.4	67	2.356	2.375	2.394	2.414	2.434	2.455	2.0
23	4245	4279	4314	4348	4383	4417	3.5	68	2.475	2.496	2.517	2.539	2.560	2.583	2.2
24	4452	4487	4522	4557	4592	4628	3.5	69	2.605	2.628	2.651	2.675	2.699	2.723	2.4
25	4663	4699	4734	4770	4806	4841	3.6	70	2.747	2.773	2.798	2.824	2.850	2.877	2.6
26	4877	4913	4950	4986	5022	5059	3.6	71	2.904	2.932	2.960	2.989	3.018	3.047	2.9
27	5095	5132	5169	5206	5243	5280	3.7	72	3.078	3.108	3.140	3.172	3.204	3.237	3.2
28	5317	5354	5392	5430	5467	5505	3.8	73	3.271	3.305	3.340	3.376	3.412	3.450	
29	5543	5581	5619	5658	5696	5735	3.8	74	3.487	3.526	3.566	3.606	3.647	3.689	
30	5773	5812	5851	5890	5930	5969	3.9	75	3.732	3.776	3.821	3.867	3.914	3.962	
31	6009	6048	6088	6128	6168	6208	4.0	76	4.011	4.061	4.113	4.165	4.219	4.275	
32	6249	6289	6330	6371	6412	6453	4.1	77	4.331	4.390	4.449	4.511	4.574	4.638	
33	6494	6536	6577	6619	6661	6703	4.2	78	4.705	4.773	4.843	4.915	4.989	5.066	
34	6745	6787	6830	6873	6916	6959	4.3	79	5.145	5.226	5.309	5.396	5.485	5.576	
35	7002	7046	7089	7133	7177	7221	4.4	80	5.671	5.769	5.871	5.976	6.084	6.197	
36	7265	7310	7355	7400	7445	7490	4.5	81	6.314	6.435	6.561	6.691	6.827	6.968	
37	7536	7581	7627	7673	7720	7766	4.6	82	7.115	7.269	7.429	7.596	7.770	7.953	
38	7813	7860	7907	7954	8002	8050	4.7	83	8.144	8.345	8.556	8.777	9.010	9.255	
39	8098	8146	8195	8243	8292	8342	4.9	84	9.514	9.788	10.08	10.39	10.71	11.06	
40	8391	8441	8491	8541	8591	8642	5.0	85	11.43	11.83	12.25	12.71	13.20	13.73	
41	8693	8744	8796	8847	8899	8952	5.2	86	14.30	14.92	15.60	16.35	17.17	18.07	
42	9004	9057	9110	9163	9217	9271	5.4	87	19.08	20.21	21.47	22.90	24.54	26.43	
43	9325	9380	9435	9490	9545	9601	5 5	88	28.64	31.24	34.37	38.19	42.96	49.10	
44	9657	9713	9770	9827	9884	9942	5.7	89	57.29	68 75	85.94	114.6	171.9	343.8	
	0	10	20′	30′	40′	50′			0′	10′	20′	30′	40′	50′	

	0	1	2	3	4	5	6	7	8	9	D
10	10000	9901	9804	9709	9615	9524	9434	9346	9259	9174	91.6
11	9091	9009	9929	8850	8772	8696	8621	8547	8475	8403	76.3
12	8333	8264	8197	8130	8065	8000	7937	7874	7813	7752	64.5
13	7692	7634	7576	7519	7463	7407	7353	7299	7246	7194	55.2
14	7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	47.9
15	6667	6623	6599	6536	6494	6452	6410	6369	6329	6289	41.9
16	6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	37.0
17	5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	32.8
18	5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	29.3
19	5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	26.4
20	5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	23.9
21	4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	21.7
22	4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	19.8
23	4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	18.2
24	4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	16.7
25	4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	15.4
26	3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	14.3
27	3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	13.3
28	3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	12.3
29	3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	11.5
30	3333	3322	3311	3301	3289	3279	3268	3257	3247	3236	10.8
31	3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	10.1
32	3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	9.5
33	3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	8.9
34	2941	2933	2924	2915	2907	2899	2890	2882	2874	2865	8.4
35	2857	2849	2841	2833	2825	2817	2809	2801	2793	2786	8.0
36	2778	2770	2762	2755	2747	2740	2732	2725	2717	2710	7.5
37	2703	2695	2688	2681	2674	2667	2660	2653	2646	2639	7.1
38	2632	2625	2618	2611	2604	2597	2591	2584	2577	2571	6.8
39	2564	2558	2551	2545	2538	2532	2525	2519	2513	2506	6.4
40	2500	2494	2488	2481	2475	2469	2463	2457	2451	2445	6.1
41	2439	2433	2427	2421	2415	2410	2404	2398	2392	2387	5.8
42	2381	2375	2370	2364	2358	2353	2347	2342	2336	2331	5.6
43	2326	2320	2315	2309	2304	2299	2294	2288	2283	2278	5.3
44	2273	2268	2262	2257	2252	2247	2242	2237	2232	2227	5.1
45	2222	2217	2212	2208	2203	2198	2193	2188	2183	2179	4.8
46	2174	2169	2165	2160	2155	2151	2146	2141	2137	2132	4.6
47	2128	2123	2119	2114	2110	2105	2101	2096	2092	2088	4.4
48	2083	2079	2075	2070	2066	2062	2058	2053	2049	2045	4.3
49	2041	2037	2033	2028	2024	2020	2016	2012	2008	2004	4.1
50	2000	1996	1992	1988	1984	1980	1976	1972	1969	1965	3.9
51	19608	9569	9531	9493	9455	9417	9380	9342	9305	9268	37.8
52	19231	9194	9157	9120	9084	9048	9011	8975	8939	8904	36.4
53	18868	8832	8797	8762	8727	8692	8657	8622	8587	8553	35.0
54	18519	8484	8450	8416	8382	8349	8315	8282	8248	8215	33.7
55	18182	8149	8116	8083	8051	8018	7986	7953	7921	7889	32.5

	0	1	2	3	4	5	6	7	8	9	D
56	17857	7825	7794	7762	7731	7699	7668	7637	7606	7575	31.4
57	17544	7513	7482	7452	7422	7391	7361	7331	7301	7271	30.3
58	17241	7212	7182	7153	7123	7094	7065	7036	7007	6978	29.3
59	16949	6920	6892	6863	6835	6807	6779	6750	6722	6694	28.3
60	16667	6639	6611	6584	6556	6529	6502	6474	6447	6420	27.4
61	16393	6367	6340	6313	6287	6260	6234	6207	6181	6155	26.5
62	16129	6103	6077	6051	6026	6000	5974	5949	5924	5898	25.6
63	15873	5848	5823	5798	5773	5748	5723	5699	5674	5649	24.8
64	15625	5600	5576	5552	5528	5504	5480	5456	5432	5408	24.1
65	15385	5361	5337	5314	5291	5267	5244	5221	5198	5175	23.3
66	15152	5129	5106	5083	5060	5038	5015	4993	4970	4948	22.6
67	14925	4903	4881	4859	4837	4815	4793	4771	4749	4728	22.0
68	14706	4684	4663	4641	4620	4599	4577	4556	4535	4514	21.3
69	14493	4472	4451	4430	4409	4388	4368	4347	4327	4306	20.7
70	14296	4265	4245	4225	4205	4184	4164	4144	4124	4104	20.1
71	14085	4065	4045	4025	4006	3986	3966	3947	3928	3908	19.6
72	13889	3870	3850	3831	3812	3793	3774	3755	3736	3717	19.0
73	13699	3680	3661	3643	3624	3605	3587	3569	3550	3532	18.5
74	13514	3495	3477	3459	3441	3423	3405	3387	3369	3351	18.0
75	13333	3316	3298	3280	3263	3245	3228	3221	3193	3175	17.6
76	13158	3141	3123	3106	3089	3072	3055	3038	3021	3004	17.1
77	12987	2970	2953	2937	2920	2903	2887	2870	2853	2837	16.7
78	12821	2804	2788	2771	2755	2739	2733	2706	2690	2674	16.2
79	12658	2642	2626	2610	2594	2579	2563	2547	2531	2516	15.8
80	12500	2484	2469	2453	2438	2422	2407	2392	2376	2361	15.4
81	12346	2330	2315	2300	2285	2270	2255	2240	2225	2210	15.1
82	12195	2180	2165	2151	2136	2121	2107	2092	2077	2063	14.7
83	12048	2034	2019	2005	1990	1976	1962	1947	1933	1919	14.4
84	11905	1891	1876	1862	1848	1834	1820	1806	1792	1779	14.0
85	11765	1751	1737	1723	1710	1696	1682	1669	1655	1641	13.7
86	11628	1614	1601	1587	1574	1561	1547	1534	1521	1507	13.4
87	11494	1481	1468	1455	1442	1429	1416	1403	1390	1377	13.1
88	11364	1351	1338	1325	1312	1299	1287	1274	1261	1249	12.8
89	11236	1223	1211	1198	1186	1173	1161	1148	1136	1123	12.5
90	11111	1099	1086	1074	1062	1050	1038	1025	1013	1001	12.2
91	10989	0977	0965	0953	0941	0929	0917	0905	0893	0881	11.9
92	10870	0858	0846	0834	0823	0811	0799	0787	0776	0764	11.7
93	10753	0741	0730	0718	0707	0695	0684	0672	0661	0650	11.4
94	10638	0627	0616	0604	0593	0582	0571	0560	0449	0537	11.2
95	10526	0515	0504	0493	0482	0471	0460	0449	0438	0428	11.0
96	10417	0406	0395	0384	0373	0363	0352	0341	0331	0320	10.7
97	10309	0299	0288	0277	0267	0256	0246	0235	0225	0214	10.5
98	10204	0194	0183	0173	0163	0152	0142	0132	0121	0111	10.3
99	10101	0091	0081	0070	0060	0050	0040	0030	0020	0010	10.1
100	10000	9900	9800	9701	9602	9502	9404	9305	9206	9108	9.9
101	99010	8912	8814	8717	8619	8522	8425	8328	8232	8135	9.7

Directions for Using the foregoing Tables.

A. To find the sine, cosine, tangent, or cotangent of any given angle.

Find from Table II., § 94, an angle between 0° and 90° whose sine or tangent will be equal to the required function. If this angle is expressed in degrees, minutes, and seconds, divide the number of seconds by six and annex the quotient to the minutes as tenths of a minute; thus: $61° 23' 44'' = 61° 23'.7 +$. Find the degrees at the left and the tens' figure in the minutes at the top of the table; thus: $\sin 61° 20' = 8774$ or $\tan 61° 20' = 1.829$. Then multiply the units and tenths of minutes by the number in column D opposite the degree; thus, in the table of sines: $1.4 \times 3.7 = 5.2$; or in the table of tangents, $1.3 \times 3.7 = 4.8$. The nearest whole number to the result is the correction, which is to be added to the sine or tangent already found. Thus, in each of the above cases the correction is 5, whence $\sin 61° 23'.7 = .8774 + 5 = .8779$; and $\tan 61° 23'.7 = 1.829 + 5 = 1.834$. [The decimal point is to be prefixed to all sines taken from the table, also to the tangents of all angles less than 45°.]

B. To find an angle whose sine or tangent is given.

Take from the table the number next less than the given number, noting the number of degrees at the left and of minutes at the top. Divide the difference between the given number and the number taken from the table, by the divisor in the column D at the right in the same horizontal line. The quotient is minutes and decimals of a minute, which are to be added to the degrees and minutes already found.

Thus, to find $\sin^{-1} .6000$. The next less sine in the table is .5995, which belongs to 36° 50′. The difference is 5, which is to be divided by 2.3 from column D. $5 \div 2.3 = 2.2$ and $36° 50' + 2'.2 = 36° 52'.2$ the required angle.

C. To find an angle whose cosine or cotangent is given.

Regard the given function as a sine or tangent, find the corresponding angle as above, and subtract the result from 90°.

Thus, to find $\cot^{-1} .4750$. The next less tangent in the table is .4734, which $= \tan 25° 20'$, and the number from column D is

3.6. $4750 - 4734 = 16$. $16 \div 3.6 = 4.4$. $25° 20' + 4'.4 = 25° 24'.4$. $90° - 25° 24'.4 = 64° 35'.6$ the required angle.

D. To find the reciprocal of a given number.

First method. — Beginning with the first significant figure of the number, find the first two figures in the column at the left, and the third figure at the top. Take the number thus found from the table, as also the number in column D on the same horizontal line. Multiply the numbers from column D by the fourth figure of the given number, with the additional figures, if any (regarded as decimals), and *subtract* the result from the number taken from the table.

Second method. — Find in the body of the table the number next greater than the given number, and, annexing ciphers to the difference, divide it by the divisor D at the right. The reciprocal of the given number will then be found as follows: The first two figures at the left, the third at the top, and the remaining figures in the quotient of the above division.

NOTE *that in the table of reciprocals a figure 1 is to be supplied before the first figure of each number in the columns headed 1, 2, 3, etc., from the line beginning 50 to and including the line beginning 99. In the two succeeding lines, which conclude the table, the figure 9 is to be supplied in the same way.*

E. The use of the table of reciprocals in the solution of problems.

When the process of division is to be performed, especially if the divisor or denominator consist of several factors, it is usually most convenient to multiply instead by the reciprocals of these factors found by Table VII. This method may be employed when a sine or cosine occurs in the denominator, but when a tangent or cotangent is a divisor the corresponding function of the complementary angle can be found from Table VI. and used as a multiplier, since the tangent and cotangent are reciprocals.

When, however, a tangent or cotangent is very large it will be more correctly obtained by finding from Table VII. the reciprocal of the complementary function than by interpolating in Table VI. Thus, the true value of tan 88° 37'.6 is 41.71; Table VI. gives 41.81; but tan 1° 22'.4 is .0240, the reciprocal of which, by Table VII., is 41.67.

To place the decimal point in a reciprocal found by either of the above methods.

If the integral part of the given number, whose reciprocal is to be found, consists of only *one* figure, then its reciprocal is entirely fractional, and the decimal point is to be put before its first figure. But if more than one figure precedes the decimal point in the given number, prefix to its reciprocal, as found from the table, as many zeros, less one, as there are figures in the integral part of the given number, and put the decimal point before the whole. Or, if the given number is a decimal fraction, with no integral part, point from the left of its reciprocal a number of figures *one more* than the number of zeros which follow the decimal point in the given number before the first significant figure.

When the first figure of a given number is 5, 6, 7, 8, or 9, the first method of finding the reciprocal will ordinarily give the most accurate result. When the first figure is 1, the second method is best. If the number begins with 2, 3, or 4, it may sometimes be expedient, for the sake of accuracy, to multiply the given number by 2 or by $\frac{1}{2}$ before finding the reciprocal, and then multiply the reciprocal found by the same factor.

The number found in column D represents the difference between successive tabular numbers in the same horizontal line with itself, and is strictly accurate only for the middle column. In interpolating between numbers in other columns, in those parts of the table where the value of D changes rapidly, the difference between two successive numbers in the table may be found by subtraction and used instead of D. The same remark applies to the table of tangents, except that in that table the difference found by subtraction must be divided by 10 for the value of D. This method must be used for the tangent of all angles greater than 72°.

Examples.

1. Find the various functions of the angle 40° 7′ 30″.

   Ans. Sin = .6445; tan = .8428; cos = .7647; cot = 1.187; cosec = 1.5516; sec = 1.3077.

2. Find the functions of 137° 24′.

   Ans. Sin = + .6769; tan = — .9195, etc.

3. Find the functions of the following angles: — 11° 35′.6; 29° 58′.1; 57° 0′ 6″; 193° 57′; 314° 30′.

4. Of what angles is .7000 the sine?

Ans. Of 44° 25′.7; of 135° 34′.3; of 404° 25′.7, etc.

Of what the cosine?

Ans. Of 45° 34′.3; of 314° 25′.7, etc.

Of what the tan? cot?

5. Proceed, as in Ex. 4, with the numbers .5125, — .8170, 2.000.

6. Divide 3 by 8.3333.

Ans. $3 \times \frac{1}{8.3333} = 3 \times .12000 = 0.36$.

MEAN ORDINATES.

112. A plane figure is a finite portion of a plane.

A plane is produced by the motion of a generating line of unlimited length upon a directrix also unlimited (§ 12). A finite portion of the generatrix, in its motion on a finite part of the directrix, describes a plane figure.

113. In the motion of the generatrix its direction remains invariable, but the length of the part, or ordinate, by which the figure is described may be either constant or changing. Thus, in Fig. 27, A is an example of a figure described with a constant ordinate, while B and C were described by variable ordinates. If a figure has been described by a variable ordinate it is important to know the mean length of the ordinate. For this purpose the portion of the directrix over which the ordinate has moved may be conceived to be divided into a number of equal parts, and an ordinate extended from each point of division. Then the sum of the lengths of these ordinates must be divided by the number of ordinates. The quotient is the mean length of the supposed number of ordinates; and if it is possible to reduce this quotient to a form equally adapted to an unlimited number of ordinates, then the true mean length of the ordinate has been found, for this is the mean of all the ordinates drawn from all the points of that

portion of the directrix over which the generatrix has passed. An example of this process will be given in § 115.

114. When such a reduction is impossible, the mean length of the ordinate can only be found approximately. See Fig. 28, where the ordinates represent the temperatures at New Haven, Connecticut, at each of the twenty-four hours of the day. The twenty-fourth part of their sum is accepted as the mean temperature of the day, though it is obvious that the true mean temperature of the day is the mean of the temperatures, not of any twenty-four instants, but of every instant through the day.

115. The mean length of the ordinate under any portion of a right line is the ordinate of its middle point, or the mean of the ordinates of its two extremities.

In Fig. 29, let $\overline{BD}$ be the given portion of a right line, let AX be the directrix, or axis of abscissas, and let $\overline{EB}$, $\overline{FC}$, $\overline{GD}$, be the ordinates, drawn to the extremities and the middle point of $\overline{BD}$. Let the distance $\overline{EF}$ be divided into any desired number of equal parts, and the distance $\overline{FG}$ into the same number, and from each point of division let ordinates extend, terminating in MN. Now, if MN have the same direction as the axis, these ordinates will all be of equal length (§ 35), and any one of them, or the mean of any two, will be equal to the mean of the whole number, however great that number be; hence, in this case, the proposition is evidently true. But if the direction of MN be different from that of AX, then, through the extremity, B, let a line pass parallel to AX. Represent the distance $\overline{EG}$ by $\mathbf{d}$ and the number of ordinates by $\mathbf{n}$, then the number of equal parts into which $\mathbf{d}$ is divided will be $\mathbf{n} - 1$, and the length of each part $\frac{1}{\mathbf{n} - 1}\mathbf{d}$. By § 37 that portion of each ordinate which is above the line BP is in a constant ratio to the distance of its lower extremity from the point B. Let this ratio be denoted by $\mathbf{r}$. Then, if the ordinate of the point B be called $\mathbf{y}$, that of the other extremity, D, must be $\mathbf{y} + \mathbf{dr}$, and the ordinate of C must be $\mathbf{y} + \frac{1}{2}\,\mathbf{dr}$. Also the length of the first ordinate, after that of B, is $\mathbf{y} + \frac{1}{\mathbf{n} - 1}\mathbf{dr}$; that of the next, $\mathbf{y} + \frac{2}{\mathbf{n} - 1}\mathbf{dr}$; the next is, $\mathbf{y} + \frac{3}{\mathbf{n} - 1}\mathbf{dr}$; and so on.

These ordinates, beginning with $\mathbf{y}$, constitute an arithmetical progression, of which the first term is $\mathbf{y}$, the common difference $\frac{1}{\mathbf{n}-1}\,\mathbf{dr}$, the number of terms $\mathbf{n}$, and the last term $\mathbf{y} + \mathbf{dr}$. By the rule of arithmetical progression, the sum of the series is $\frac{\mathbf{n}}{2}\,(2\,\mathbf{y} + \mathbf{dr})$, and, dividing this sum by $\mathbf{n}$, the mean length of the ordinates is found to be $\frac{1}{2}\,(2\,\mathbf{y} + \mathbf{dr})$ or $\mathbf{y} + \frac{1}{2}\,\mathbf{dr}$. By inspection of this result we see, *first*, that it is independent of $\mathbf{n}$, and hence applicable to any number of ordinates without limit; hence the true mean ordinate has been found; *second*, it is seen that this mean ordinate is identical with the ordinate of the middle point, C, which had been previously found to be $\mathbf{y} + \frac{1}{2}\,\mathbf{dr}$; and, *third*, the mean ordinate is half the sum of the ordinates of the extremities, for $\mathbf{y} + \frac{1}{2}\,\mathbf{dr} = \frac{1}{2}\,(\mathbf{y} + \overline{\mathbf{y} + \mathbf{dr}})$.

AREA.

116. The area of a figure is the amount of surface it contains.

117. The area of any plane figure described with a variable ordinate is equal to that of the figure which would have been described by the same generatrix, moving to the same distance on the directrix, but with a constant ordinate equal to the mean ordinate.

Thus, in Fig. 28, if 3M be the mean ordinate of the curve BCD, then, in moving to the distance AG on the directrix, it will pass over as much surface as does the variable ordinate of the curve, for its excess of length in any one portion of its path is exactly balanced by its deficiency in another.

118. Let two figures be described by the same generatrix and directrix; then, if the distances over which the generatrix has moved be equal in each, their areas will be as their mean ordinates; but if their mean ordinates are equal, then their areas are as the distances on the directrix over which these ordinates have moved: in general, their areas are as

the product of their mean ordinates into the distances on the directrix over which they have respectively moved.

In Fig. 30 let the length of the ordinate $\overline{AE}$ or $\overline{BF}$ be unity, while that of $\overline{BG}$ is $\mathbf{n}$; it is plain that when the generatrix has moved over a distance, $\overline{AB}$ or $\overline{BC}$, equal to unity, then the area passed over by $\overline{AE}$ (the figure I) will be to that passed over by $\overline{BG}$ (the figure K) as $\mathbf{n}$ to 1. Also, that when the generatrix has moved over a distance $\overline{CD}$ or $\mathbf{m}$, the area of the figure R will be to that of K as $\mathbf{m}$ to 1; hence the area of R is to that of I as $\mathbf{mn}$ to 1. Moreover, if any other ordinate $\mathbf{q}$ of the same generatrix moves over the distance $\mathbf{p}$ of the directrix, the area generated will for the same reason be to I as $\mathbf{pq}$ to 1, hence it will be to R as $\mathbf{pq}$ to $\mathbf{mn}$.

119. Let two figures be described by different generatrices and directrices; then, if each be described by a unit-ordinate moving to a unit's distance on the directrix, their areas will be as the sines of the angles which their respective generatrices make with the directrices.

In Fig. 31 let G and H be two such figures, their directrices having been placed so as to coincide. Through any point, B, of this directrix let a line pass parallel to the generatrix, AY, and through any other point, C, a line parallel to the other generatrix, AY′, and intersecting the former line in D. Let the distance $\overline{BC}$ on the directrix be denoted by $\mathbf{c}$, and the distances $\overline{CD}$ and $\overline{BD}$ by $\mathbf{a}$ and $\mathbf{b}$ respectively. Also let the angle made with AX by AY be called α, and that made by A′Y′ be called β. Now, the figure E, included by the lines BD, CD, and AX, may be considered as described by the same generatrix as G, the mean ordinate being $\frac{\mathbf{b}}{2}$ (§ 115), hence the proportion

$$\text{area E} : \text{area G} :: \frac{\mathbf{bc}}{2} : 1.$$

But the figure E may also be considered as described by the same generatrix as H, the mean ordinate being $\frac{\mathbf{a}}{2}$;

hence $$\text{area E} : \text{area H} :: \frac{\mathbf{ac}}{2} : 1.$$

Since the extremes in the two proportions are alike, the means are inversely proportional, whence

$$\textbf{area } G : \textbf{area } H :: \frac{\mathbf{ac}}{2} : \frac{\mathbf{bc}}{2}.$$

But $$\frac{\mathbf{ac}}{2} : \frac{\mathbf{bc}}{2} :: \mathbf{a} : \mathbf{b},$$

and $$\mathbf{a} : \mathbf{b} :: \sin\alpha : \sin\beta \qquad (\S\ 53).$$

Therefore $$\textbf{area } G : \textbf{area } H :: \sin\alpha : \sin\beta,$$

which was to be proved.

120. *The unit of area* is the amount of surface described by a unit-ordinate, passing over a unit's distance on the directrix, when the sine of the angle of the generatrix and directrix is also unity; i.e., when those lines are perpendicular to each other.

121. The numerical expression for the area of any plane figure is the continued product of the mean ordinate with which it is described, the distance on the directrix over which the ordinate moves in describing it, and the sine of the angle which the moving line makes with the directrix.

For in Fig. 30 it has been shown that the figure R is to I as $\mathbf{mn}$ is to 1, but (§ 119) the area of I is to the unit of area as $\sin\omega$ is to 1, hence

$$\text{area } R : 1 :: \mathbf{mn} \sin\omega : 1,$$

or $$\text{area} = \mathbf{mn} \sin\omega.$$

122. The area under that part of a right line included between any two points $\mathbf{x}'$, $\mathbf{y}'$, and $\mathbf{x}''$, $\mathbf{y}''$, is

$$\tfrac{1}{2}(\mathbf{x}'' - \mathbf{x}')(\mathbf{y}'' + \mathbf{y}') \sin\omega,$$

where ω represents the angle of the axes.

It is here assumed that the axis of abscissas is the directrix, and that the axis of ordinates has the direction of the generatrix. The theorem is but an application of that of § 121, for the mean ordinate is $\frac{1}{2}(\mathbf{y}'' + \mathbf{y}')$ (§ 115), and the distance to which it moves on the axis of abscissas is plainly $\mathbf{x}'' - \mathbf{x}'$.

Examples.

In each of the following examples the coördinates of two points

are given, to find the area under the rectilinear distance between them. In each case the student should not only compute the area but construct the figure.

I.—When $\omega = 90°$.

Coördinates.	Ex. 1st.	2d.	3d.	4th.	5th.	6th.	7th	8th.	9th.
$x' =$	3	2	—3	—5	1	—2	—9	5	—7
$y' =$	2	7	3	4	—2	—8	3	11	—11
$x'' =$	9	6	2	0	7	5	—2	5	—1
$y'' =$	5	0	8	4	5	—1	—3	3	—3

II.—When $\omega = 60°$. (§ 108.) $\omega = 150°$.

Coördinates.	Ex. 10th.	11th.	12th.	13th.
$x' =$	3	0	2	0
$y' =$	5	8	4	$8\sqrt{3}$
$x'' =$	7	2	5	6
$y'' =$	5	4	—2	0

In example 5th and some of those which follow, one of the ordinates is negative while the other is positive. In this case the area computed will be the algebraic sum of two parts, a positive and a negative area, one lying above and the other below the axis. As these two parts are of opposite signs, their algebraic sum is of course their arithmetical difference. If the negative ordinate be the larger, or if both ordinates be negative, then the computed area will be negative unless the abscissas are taken in such an order as to make $x'' - x'$ negative.

123. Two plane figures are said to be *similar* to one another when they differ only in the scale of magnitude upon which each is constructed; or, in other words, when the distance between any two points in one figure is in a constant ratio to the distance between corresponding points in the other figure.

For example, two maps or plans of the same piece of ground, drawn to different scales, are similar figures.

If r denote the ratio of homologous distances (that is, of the distances between corresponding points) in two similar figures,

then, since the ordinates by which the two figures are described must move under like laws of inclination to the directrix and variation in length, the mean ordinate **n** of the first figure will correspond to a mean ordinate **rn** of the second; while if the distances over which the ordinate of the first figure moves be **m**, the corresponding distance in the second figure will be **rm**. Therefore, the area of the former figure will be to that of the latter in the ratio of **mn** to **rm** × **rn**, or of 1 to $\mathbf{r}^2$. Hence,

The areas of similar figures are to each other in the duplicate ratio of their homologous distances.

POLYGONS.

124. The boundary-line which separates any plane figure from the adjacent surface is called its *periphery*. The periphery may consist of a continuous curve, or it may be in part curved and in part straight, like C in Fig. 27, or it may consist of a broken line having each of its parts straight. In the last case the figure is called a *polygon*. The periphery of a polygon is also called its *perimeter*. The straight portions of the broken line which constitutes the perimeter are severally called *sides* of the polygon, and the intersections in which the sides terminate are called *vertices*.

125. A polygon of three sides is called a *triangle;* of four sides, a *quadrilateral;* of five sides, a *pentagon;* of six, a *hexagon*, etc. A quadrilateral having no two sides parallel is called a *trapezium;* if two sides are parallel, a *trapezoid;* if the four sides are parallel, two and two, a *parallelogram*. In a parallelogram the opposite sides are equal to each other (§ 35). If *adjacent* sides are equal to each other, the parallelogram is a *rhomb;* if perpendicular to each other, it is a *rectangle;* if they are neither equal nor perpendicular, a *rhomboid;* if both equal and perpendicular, a *square*.

126. In a triangle, or in any kind of a parallelogram, any one of the sides may be taken as the *base*. In a trapezoid, either of the parallel sides may be taken as the base. Then, any vertex except the two extremities of the base may be taken as the *prin-*

cipal vertex of the figure. The angle which a side extending from the principal vertex to the base makes with the base is called the *base angle.* The angle between the two sides which meet in the principal vertex is called the *vertical angle.* The distance from the principal vertex to the base, in a direction perpendicular to the base, is called the *altitude.* The altitude is equal to the product of the sine of the base angle into the side extending from the principal vertex to the base.

127. The areas of triangles, trapezoids, and parallelograms may be found by a simple application of the rule for the area under a right line. The base is to be taken for the axis of ordinates, and the adjacent side extending to the principal vertex, for the axis of abscissas. Then, in the parallelogram, the mean ordinate is the base, in the triangle half the base, and in the trapezoid half the sum of the parallel sides. [The student will give the reason in each case, and will also deduce the following propositions:]

The area of the trapezoid is the product of the altitude into half the sum of the parallel sides, or into the distance between the middle points of the sides which are not parallel.

The area of the triangle is half the product of the altitude and base, or half the product of two adjacent sides into the sine of the angle which they make with each other.

The area of the parallelogram is the product of the altitude and base, or the product of any two adjacent sides into the sine of their angle.

The area of the rectangle is the product of any two adjacent sides.

The area of the square is the second power of any one of its sides.

[The last proposition indicates the origin of the use of the word "square" as a synonym for "second power."]

128. To find the area of a trapezium, or of any polygon of more than four sides, it is sometimes most convenient to divide the polygon into parts by *diagonal lines*, and compute the area of the parts separately. But when the coördinates of the vertices are given, or can be found, the following rule may be used:

Find the area under each side separately; the algebraic sum of these areas will be the area of the polygon.

Observe that as the formula $\frac{1}{2}(\mathbf{x}'' - \mathbf{x}')(\mathbf{y}'' + \mathbf{y}') \sin \omega$ is applied to the successive sides, the coördinates of each vertex appear in *two* of the separate computations and no more, being substituted in one of these computations for $\mathbf{x}''$, $\mathbf{y}''$, and in the other for $\mathbf{x}'$, $\mathbf{y}'$.

Thus, in Fig. 32, beginning at the vertex C, and proceeding toward the right, the area under the first side is positive; but as the abscissa of the vertex E is less than that of D, the area under this side, $\overline{DE}$, is negative, because the factor $\mathbf{x}'' - \mathbf{x}'$ is so; and so with all the succeeding areas until the vertex G is reached. Now, if the sum of all the negative areas be (algebraically) added to that of the positive areas, the entire sum (or arithmetical difference of the partial sums) will be the area of the polygon, which, in this case, is negative.

By drawing figures to illustrate all of the following examples, the student will convince himself that negative coördinates of some of the vertices will not affect the truth of the result.

Coördinates of Vertices.

Example.	1st.		2d.		3d.		4th.		5th.		6th.	
	x	y	x	y	x	y	x	y	x	y	x	y
1st . .	19,	0	23,	3	11,	8	7,	5				
2d . .	9,	—5	13,	—2	1,	3	— 3,	0				
3d . .	0,	0	—6,	—8	—11,	—3	—12,	—10	—4,	—12	+3,	—9
4th . .	—2,	0	6,	0	6,	5						
5th . .	13,	1	9,	7	27,	12						
6th . .	9,	7	6,	6	1,	19						
7th . .	13,	1	9,	7	6,	6	1,	19	9	7	27,	12
8th . .	13,	1	1,	19	6,	6	27,	12				

129. The area of the triangle whose vertices are $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{x}''$, $\mathbf{y}''$, and $\mathbf{x}'''$, $\mathbf{y}'''$, is

$$\tfrac{1}{2} \sin \omega \; [(\mathbf{x}'' - \mathbf{x}')(\mathbf{y}'' + \mathbf{y}') + (\mathbf{x}''' - \mathbf{x}'')(\mathbf{y}''' + \mathbf{y}'') + (\mathbf{x}' - \mathbf{x}''')(\mathbf{y}' + \mathbf{y}''')];$$

or, expanding and rearranging terms,

$$\text{area} = \tfrac{1}{2} \sin \omega \; [\mathbf{y}'(\mathbf{x}'' - \mathbf{x}''') + \mathbf{y}''(\mathbf{x}''' - \mathbf{x}') + \mathbf{y}'''(\mathbf{x}' - \mathbf{x}'')].$$

130. The process of § 128 is much used in surveying. This application will be understood from an example (Ex. 1st, below). In surveying the field (Fig. 32) the surveyor begins at the vertex C, and goes entirely around the field, measuring the length of each side, and the angle which it makes either with a meridian or with some line that passes through the vertex C and is assumed as an axis of abscissas. These measurements he tabulates as in the first two columns of the table below, on page 62. The next two columns contain the cosine and sine respectively of the angles in the second column. The next columns, headed $\mathbf{x}''-\mathbf{x}'$ and $\mathbf{y}''-\mathbf{y}'$, are computed by multiplying the distances from the first column into the corresponding cosines and sines. (When the axis, AX, is a meridian, — north and south line, — this difference of like coördinates of the two extremities of a side is called the northing or southing and the easting or westing of the side.) The algebraic sums of these two columns separately should each be zero, as the student may readily show, and, when such a result is obtained, it is a proof of the correctness of the measurement and of the computation thus far. In the present instance, in consequence of some slight error of measurement, the sum of the column $\mathbf{x}''-\mathbf{x}'$ is .0001 and that of $\mathbf{y}''-\mathbf{y}'$ is $-.0003$. When such errors are not sufficiently large to demand a remeasurement of the field, the error is usually *distributed* to the different sides in proportion to their length. The succeeding column is the result of correcting in this way the column $\mathbf{y}''-\mathbf{y}'$. In the next column, headed $\mathbf{y}''$, the first number is transferred from the preceding column, while each succeeding number is the sum of the number just above it and that at the left of it. In the next column, headed $\mathbf{y}''+\mathbf{y}'$, the first number is transferred as before, while each of those which follow is the sum of two numbers in the column headed $\mathbf{y}''$, namely, the one in the same horizontal line, and that in the line above. Two more columns are now formed by multiplying the numbers from the column $\mathbf{x}''-\mathbf{x}'$ into the corresponding numbers in the column $\mathbf{y}''+\mathbf{y}'$, and placing the positive and negative products in separate columns. Finally, each of these last two columns is added, the less sum subtracted (algebraically added) from the greater, and the difference divided by 2. The result, neglecting its sign, is the area of the field.

EXAMPLE 1.

Sides.	Angles.	Cosines	Sines.	$x''-x'$	$y''-y'$	$y''-y'$ (corr.)	y''	$y''+y'$	+ Areas.	—Areas.	
7.20	41° 27′	+.7495	+.6620	+5.3966	+4.7662	+4.7663	+ 4.7663	+ 4.7663	+25.7218		172.4442 37.4487
8.35	120° 35′	—.5088	+.8609	—4.2484	+7.1884	+7.1885	+11.9548	+16.7211		—71.0379	2)134.9955 67.4978
4.62	208° 12′	—.8813	—.4726	—4.0716	—2.1832	—2.1832	+ 9.7716	+21.7264		—88.4612	
6.87	261° 30′	—.1478	—.9890	—1.0154	—6.7945	—6.7944	+ 2.9772	+12.7488		—12.9451	6.74978 A
$4.93\frac{3}{4}$	322° 55′	+.7978	—.6030	+3.9389	—2.9772	—2.9772	0	+ 2.9772	+11.7269		
									37.4487	172.4442	

In the above example, if the unit with which the sides were measured was the chain of 66 feet, the area 67.4978 is expressed in square chains; and since ten of these make an acre, the area in acres and decimals is found by moving the decimal point one place.

EXAMPLE 2. — The measured lengths of the sides of a field are 6.26 ch., 7.10 ch., 4.44 ch., 2.58 ch., 5.40 ch.; and the angles which they severally make with the meridian are 56° 30′, 145° 45′, 237° 0′, 276° 30′, and 325°. Required the area of the field.

PLANE TRIGONOMETRY.

131. Plane Trigonometry has for its object the *solution* of plane triangles, by which is meant, computing from certain known *parts*, (sides or angles,) the unknown parts of a triangle and its area. From any three of the six parts of a triangle, except the three angles, the other parts and the area may be found.

132. In the usage of trigonometry, the interior angle which any two sides make with each other is said to be *adjacent* to either of these sides, *included* by both of them, and *opposite* to the third. In the following §§, the sides of a triangle will be denoted by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the angles opposite each respectively by α, β, and γ, and the area by A.

133. All formulæ for the solution of triangles are based on the following, with which the student is already familiar:

I. $\alpha + \beta + \gamma = 180°$. (§ 29.)

II. $\mathbf{a} : \mathbf{b} :: \sin\alpha : \sin\beta$;
$\mathbf{a} : \mathbf{c} :: \sin\alpha : \sin\gamma$;
$\mathbf{b} : \mathbf{c} :: \sin\beta : \sin\gamma$. (§ 53.)

III. $A = \frac{1}{2}\,\mathbf{bc}\sin\alpha = \frac{1}{2}\,\mathbf{ac}\sin\beta = \frac{1}{2}\,\mathbf{ab}\sin\gamma$. (§ 127.)

134. In special cases, however, the foregoing formulæ need to be supplemented by certain others which belong merely to the theory of angles, and might have been presented earlier but that they would have been too widely separated from their application. They are deduced as follows:

Let β and γ be any two angles, then

$$\beta = \tfrac{1}{2}(\beta + \gamma) + \tfrac{1}{2}(\beta - \gamma) \quad \text{and} \quad \gamma = \tfrac{1}{2}(\beta + \gamma) - \tfrac{1}{2}(\beta - \gamma).$$

Hence (§ 101)

$$\sin\beta = \sin\tfrac{1}{2}(\beta + \gamma)\cos\tfrac{1}{2}(\beta - \gamma) + \cos\tfrac{1}{2}(\beta + \gamma)\sin\tfrac{1}{2}(\beta - \gamma),$$

and

$\sin\gamma = \sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\gamma) - \cos\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\beta-\gamma)$.

Hence

$$\sin\beta + \sin\gamma = 2\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta-\gamma) \qquad (1)$$

and

$$\sin\beta - \sin\gamma = 2\cos\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\beta-\gamma) \qquad (2)$$

also (§ 102)

$$\sin(\beta+\gamma) = 2\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta+\gamma). \qquad (3)$$

From (1) and (2)

$$\sin\beta + \sin\gamma : \sin\beta - \sin\gamma :: 2\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta-\gamma) : 2\cos\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\beta-\gamma);$$

whence, dividing the terms of the second ratio by $2\cos\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\gamma)$, and remembering that $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha$,

$$\sin\beta + \sin\gamma : \sin\beta - \sin\gamma :: \tan\tfrac{1}{2}(\beta+\gamma) : \tan\tfrac{1}{2}(\beta-\gamma). \qquad (4)$$

More simply, from (1) and (3),

$$\sin\beta + \sin\gamma : \sin(\beta+\gamma) :: \cos\tfrac{1}{2}(\beta-\gamma) : \cos\tfrac{1}{2}(\beta+\gamma) \qquad (5)$$

and from (2) and (3),

$$\sin\beta - \sin\gamma : \sin(\beta+\gamma) :: \sin\tfrac{1}{2}(\beta-\gamma) : \sin\tfrac{1}{2}(\beta+\gamma) \qquad (6)$$

135. When any two angles of a triangle are given, the third may be found by Formula I., § 133. The process is so simple that all the angles may be considered as given if any two are, hence the possible cases which may arise in the solution of triangles are reduced to the four following:

I. Given the angles and a side; as, given α, β, γ, and **a**, to find **b**, **c**, and A.

II. Given two sides and an angle opposite one; as, given α with **a** and **b**, to find β, γ, **c**, and A.

III. Given two sides and the included angle; as, given α with **b** and **c**, to find β, γ, **a**, and A.

IV. Given the three sides, **a**, **b**, and **c**, to find α, β, γ, and A.

136. Case I. — Given α, β, γ, and **a**.

It is simplest to find one of the unknown sides before finding the area. The formulæ are, — (see § 133, II. and III.)

$$\sin\alpha : \sin\beta :: \mathbf{a} : \mathbf{b}$$

$$\sin\alpha : \sin\gamma :: \mathbf{a} : \mathbf{c}$$
$$\tfrac{1}{2}\,\mathbf{ab}\sin\gamma \quad \text{or} \quad \tfrac{1}{2}\,\mathbf{ac}\sin\beta = \mathrm{A}.$$

If it be desired to find the area directly from the given parts, the following formula may easily be deduced from the foregoing:

$$\frac{\mathbf{a}^2\sin\beta\sin\gamma}{2\sin\alpha} = \mathrm{A}.$$

137. CASE II. — Given α, **a** and **b**.

This case is reduced to Case I. by finding the unknown angles. The formulæ are:

$$\mathbf{a} : \mathbf{b} :: \sin\alpha : \sin\beta$$
$$180° - (\alpha + \beta) = \gamma.$$

In finding the angle β from the table when its sine has been computed, the student will remember that the same sine belongs to several angles, for which see Table II. § 94. But no negative value of β must be taken, nor any positive value which, substitued in the formula $180° - (\alpha + \beta) = \gamma$, will render γ negative. No more than two values can possibly satisfy this test, but by applying it in each particular example the student will find whether the conditions of that example may be met by two different triangles, or by only one. When there are two, the values of β in the two triangles are supplements of each other.

138. CASE III. — Given α, **b**, and **c**.

The area may be found directly by the formula (§ 133, III.)

$$\tfrac{1}{2}\,\mathbf{bc}\sin\alpha = \mathrm{A}.$$

To find the unknown angles the proportion (§ 133, II.)

$$\mathbf{b} : \mathbf{c} :: \sin\beta : \sin\gamma$$

is to be taken by composition and division:

$$\mathbf{b} + \mathbf{c} : \mathbf{b} - \mathbf{c} :: \sin\beta + \sin\gamma : \sin\beta - \sin\gamma.$$

But, (§ 134, Equation 4)

$$\sin\beta + \sin\gamma : \sin\beta - \sin\gamma :: \tan\tfrac{1}{2}(\beta + \gamma) : \tan\tfrac{1}{2}(\beta - \gamma).$$

Hence $\quad \mathbf{b} + \mathbf{c} : \mathbf{b} - \mathbf{c} :: \tan\tfrac{1}{2}(\beta + \gamma) : \tan\tfrac{1}{2}(\beta - \gamma).$

In this proportion, the first and second terms are very easily found, as is also the third, since $\beta + \gamma = 180° - \alpha$. Hence the

fourth term is determined, and β and γ may be obtained from the formulæ:

$$\tfrac{1}{2}(\beta+\gamma)+\tfrac{1}{2}(\beta-\gamma)=\beta \qquad \tfrac{1}{2}(\beta+\gamma)-\tfrac{1}{2}(\beta-\gamma)=\gamma.$$

The angles being found, this case is reduced to case I.

139. Case IV. — Given **a**, **b**, and **c**.

Since $\alpha=180°-(\beta+\gamma)$, $\sin\alpha=\sin(\beta+\gamma)$ (§ 89), whence (§ 133, II.)

$$\sin(\beta+\gamma) : \sin\beta :: \mathbf{a} : \mathbf{b} \qquad (1).$$

Taking the proportion $\sin\beta : \sin\gamma :: \mathbf{b} : \mathbf{c}$ by composition and then by division, we have

$$\sin\beta+\sin\gamma : \sin\beta :: \mathbf{b}+\mathbf{c} : \mathbf{b} \qquad (2)$$

$$\sin\beta-\sin\gamma : \sin\beta :: \mathbf{b}-\mathbf{c} : \mathbf{b} \qquad (3)$$

From (1) and (2)

$$\sin\beta+\sin\gamma : \sin(\beta+\gamma) :: \mathbf{b}+\mathbf{c} : \mathbf{a} \qquad (4)$$

From (1) and (3)

$$\sin\beta-\sin\gamma : \sin(\beta+\gamma) :: \mathbf{b}-\mathbf{c} : \mathbf{a} \qquad (5)$$

Combining proportions (4) and (5) of the present § with (5) and (6) of § 134 by equality of ratios,

$$\cos\tfrac{1}{2}(\beta-\gamma) : \cos\tfrac{1}{2}(\beta+\gamma) :: \mathbf{b}+\mathbf{c} : \mathbf{a} \qquad (6)$$

and $\sin\tfrac{1}{2}(\beta-\gamma) : \sin\tfrac{1}{2}(\beta+\gamma) :: \mathbf{b}-\mathbf{c} : \mathbf{a} \qquad (7)$

But $\tfrac{1}{2}(\beta+\gamma)=\tfrac{1}{2}(180°-\alpha)=90°-\tfrac{1}{2}\alpha,$

hence $\cos\tfrac{1}{2}(\beta+\gamma)=\sin\tfrac{1}{2}\alpha,$

and $\sin\tfrac{1}{2}(\beta+\gamma)=\cos\tfrac{1}{2}\alpha.$

Substituting these values in the proportions (6) and (7), and converting them to equations, they become:

$$\cos\tfrac{1}{2}(\beta-\gamma)=\frac{\mathbf{b}+\mathbf{c}}{\mathbf{a}}\sin\tfrac{1}{2}\alpha \qquad (8)$$

$$\sin\tfrac{1}{2}(\beta-\gamma)=\frac{\mathbf{b}-\mathbf{c}}{\mathbf{a}}\cos\tfrac{1}{2}\alpha \qquad (9)$$

Squaring these values of $\cos\tfrac{1}{2}(\beta-\gamma)$ and $\sin\tfrac{1}{2}(\beta-\gamma)$ and adding the results, we have, because the sum of the squares of the sine and cosine of any angle is unity (§ 48)

$$1=\frac{(\mathbf{b}+\mathbf{c})^2}{\mathbf{a}^2}\sin^2\tfrac{1}{2}\alpha+\frac{(\mathbf{b}-\mathbf{c})^2}{\mathbf{a}^2}\cos^2\tfrac{1}{2}\alpha. \qquad (10)$$

This equation may be put in a form containing only one function of $\frac{1}{2}a$, either by substituting

$1 - \cos^2 \frac{1}{2}a$ for $\sin^2 \frac{1}{2}a$ or $1 - \sin^2 \frac{1}{2}a$ for $\cos^2 \frac{1}{2}a$.

The latter substitution gives

$$1 = \frac{(\mathbf{b}+\mathbf{c})^2}{\mathbf{a}^2} \sin^2 \tfrac{1}{2}a + \frac{(\mathbf{b}-\mathbf{c})^2}{\mathbf{a}^2} - \frac{(\mathbf{b}-\mathbf{c})^2}{\mathbf{a}^2} \sin^2 \tfrac{1}{2}a,$$

whence
$$\frac{4\,\mathbf{bc}}{\mathbf{a}^2} \sin^2 \tfrac{1}{2}a = 1 - \frac{(\mathbf{b}-\mathbf{c})^2}{\mathbf{a}^2}$$

or
$$\sin^2 \tfrac{1}{2}a = \frac{\mathbf{a}^2 - (\mathbf{b}-\mathbf{c})^2}{4\,\mathbf{bc}}. \tag{11}$$

The numerator of this fraction may be factored, and written $(\mathbf{a}-\mathbf{b}+\mathbf{c})\,(\mathbf{a}+\mathbf{b}-\mathbf{c})$; and if $\mathbf{s}$ be put for $\mathbf{a}+\mathbf{b}+\mathbf{c}$ the fraction becomes $\dfrac{(\mathbf{s}-2\,\mathbf{b})\,(\mathbf{s}-2\,\mathbf{c})}{4\,\mathbf{bc}}$, or, dividing both numerator and denominator by 4

$$\sin^2 \tfrac{1}{2}a = \frac{\left(\frac{\mathbf{s}}{2}-\mathbf{b}\right)\left(\frac{\mathbf{s}}{2}-\mathbf{c}\right)}{\mathbf{bc}},$$

whence extracting the square root

$$\sin \tfrac{1}{2}a = \sqrt{\frac{\left(\frac{\mathbf{s}}{2}-\mathbf{b}\right)\left(\frac{\mathbf{s}}{2}-\mathbf{c}\right)}{\mathbf{bc}}}. \tag{12}$$

By pursuing a similar line of reduction from the point where $1 - \cos^2 \frac{1}{2}a$ is put for $\sin^2 \frac{1}{2}a$ in equation (10), the student will obtain

$$\cos \tfrac{1}{2}a = \sqrt{\frac{\frac{\mathbf{s}}{2}\left(\frac{\mathbf{s}}{2}-\mathbf{a}\right)}{\mathbf{bc}}}. \tag{13}$$

Dividing (11) by (12) (see § 49),

$$\tan \tfrac{1}{2}a = \sqrt{\frac{\left(\frac{\mathbf{s}}{2}-\mathbf{b}\right)\left(\frac{\mathbf{s}}{2}-\mathbf{c}\right)}{\frac{\mathbf{s}}{2}\left(\frac{\mathbf{s}}{2}-\mathbf{a}\right)}}. \tag{14}$$

If equation (11) be multiplied through by 2, and then each member subtracted from unity the first member, $1 - 2\sin^2 \frac{1}{2}a$ becomes equal to $\cos a$, since $\cos 2a = 1 - 2\sin^2 a$ (§ 102).

Hence
$$\cos a = 1 - \frac{\mathbf{a}^2 - (\mathbf{b}-\mathbf{c})^2}{2\,\mathbf{bc}}$$

or
$$\cos a = \frac{\mathbf{b}^2 + \mathbf{c}^2 - \mathbf{a}^2}{2\,\mathbf{bc}}. \tag{15}$$

Multiplying (12) and (13) together, and the product by 2, remembering that $2 \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha = \sin \alpha$ (§ 102), we have

$$\sin \alpha = \frac{2}{\mathbf{bc}} \sqrt{\frac{\mathbf{s}}{2}\left(\frac{\mathbf{s}}{2}-\mathbf{a}\right)\left(\frac{\mathbf{s}}{2}-\mathbf{b}\right)\left(\frac{\mathbf{s}}{2}-\mathbf{c}\right)}. \qquad (16)$$

And substituting this value of $\sin \alpha$ in the formula $A = \frac{1}{2} \mathbf{bc} \sin \alpha$;

$$A = \sqrt{\frac{\mathbf{s}}{2}\left(\frac{\mathbf{s}}{2}-\mathbf{a}\right)\left(\frac{\mathbf{s}}{2}-\mathbf{b}\right)\left(\frac{\mathbf{s}}{2}-\mathbf{c}\right)}. \qquad (17)$$

Equation (17) is the formula for finding the area directly from the given sides, The angle α may be found by using either of the equations (12), (13), (14), (15), or (16); but of these (14) will usually be found the most convenient. When one angle has been found, the others may be computed by the methods of preceding cases, or, better, the process of finding the first angle may be repeated, using the analogous formulæ,

$$\tan \tfrac{1}{2} \beta = \sqrt{\frac{\left(\frac{\mathbf{s}}{2}-\mathbf{a}\right)\left(\frac{\mathbf{s}}{2}-\mathbf{c}\right)}{\frac{\mathbf{s}}{2}\left(\frac{\mathbf{s}}{2}-\mathbf{b}\right)}},$$

and

$$\tan \tfrac{1}{2} \gamma = \sqrt{\frac{\left(\frac{\mathbf{s}}{2}-\mathbf{a}\right)\left(\frac{\mathbf{s}}{2}-\mathbf{b}\right)}{\frac{\mathbf{s}}{2}\left(\frac{\mathbf{s}}{2}-\mathbf{c}\right)}}.$$

140. In the first three of the foregoing cases, the solution is much simplified when the value of a known angle is 90°. The formulæ for solution, which are here appended for the sake of completeness, may in that case be derived directly from the definitions of the functions of angles. The student should give the proof in detail, and may also reduce the formulæ of the preceding §§, by substituting 90° for one of the known angles. In the following formulæ it is assumed that the angle α is a right angle.

Case I. — Given β and $\mathbf{a}$.

$\gamma = 90° - \beta \qquad \mathbf{b} = \mathbf{a} \sin \beta. \qquad \mathbf{c} = \mathbf{a} \cos \beta.$

Case II. — Given β and $\mathbf{b}$.

$\gamma = 90° - \beta.$ $\quad \mathbf{a} = \dfrac{\mathbf{b}}{\sin \beta}.$ $\quad \mathbf{c} = \mathbf{b} \cot \beta.$

CASE III. — Given β and $\mathbf{c}$.

$\gamma = 90° - \beta.$ $\quad \mathbf{a} = \dfrac{\mathbf{c}}{\cos \beta}.$ $\quad \mathbf{b} = \mathbf{c} \tan \beta.$

CASE IV. — Given $\mathbf{a}$ and $\mathbf{b}$.

$\sin \beta = \dfrac{\mathbf{b}}{\mathbf{a}}.$ $\quad \cos \gamma = \dfrac{\mathbf{b}}{\mathbf{a}}.$ $\quad \mathbf{c} = \sqrt{(\mathbf{a}+\mathbf{b})(\mathbf{a}-\mathbf{b})}.$

CASE V. — Given $\mathbf{b}$ and $\mathbf{c}$.

$\tan \beta = \dfrac{\mathbf{b}}{\mathbf{c}}.$ $\quad \tan \gamma = \dfrac{\mathbf{c}}{\mathbf{b}}.$ $\quad \mathbf{a} = \sqrt{\mathbf{b}^2 + \mathbf{c}^2},$ or

$$\mathbf{a} = \frac{\mathbf{b}}{\sin \beta}.$$

EXAMPLES.

141. (A figure should be drawn for each.)

1. Given $\mathbf{a} = 1760$, $\beta = 85° 15'$, $\gamma = 83° 45'$.
2. Given $\mathbf{a} = 5$, $\mathbf{b} = 5$, $\alpha = 30°$. (See Fig. 33.)
3. Given $\beta = 84° 47' 38''$, $\gamma = 40° 10'$, $\mathbf{a} = 145$.
4. Given $\mathbf{b} = 5780$, $\mathbf{c} = 7639$, $\beta = 43° 8'$.
5. Given $\mathbf{a} = 37$, $\mathbf{b} = 13$, $\mathbf{c} = 40$.
6. Given $\mathbf{a} = 445$, $\mathbf{b} = 83$, $\gamma = 87° 55'$.
7. Given $\mathbf{a} = 73$, $\mathbf{b} = 55$, $\alpha = 90°$.
8. Given $\mathbf{a} = 212.5$, $\mathbf{b} = 836.4$, $\alpha = 14° 24' 25''$.
9. Given $\mathbf{c} = 520$, $\gamma = 66° 2' 52''$, $\mathbf{a} = 569$.
10. Given $\alpha = 90°$, $\mathbf{a} = 401$, $\mathbf{b} = 399$.
11. Given $\mathbf{a} = 353$, $\mathbf{b} = 272$, $\mathbf{c} = 225$.
12. Given $\alpha = 101° 17'$, $\beta = 78° 43'$, $\mathbf{c} = 125$.

What absurdity is involved in the statement of this problem?

13. Given $\alpha = 67° 22' 49''$, $\beta = 45° 14' 22''$, $\mathbf{b} = 100$.

Show from the formulæ used in this example, that if two angles of a triangle are equal, the sides opposite them are also equal.

14. Given $\mathbf{a} = 79$, $\mathbf{b} = 31.6$, $\beta = 98° 53'$.

This example illustrates the proposition that the greatest angle of any triangle is always opposite the greatest side, which may be

proved as follows: — Let β be the greatest angle, and γ either of the others; then $\sin\beta > \sin\gamma$, whether β be greater or less than 90°, for as angles increase from 0° to 90°, the sine increases uninterruptedly; hence if $\beta < 90°$, then β and γ are two angles each less than 90° of which β is the greater; hence $\sin\beta > \sin\gamma$. But if $\beta > 90°$, then $180° - \beta < 90°$; and since $180° - \beta = \alpha + \gamma$, $180° - \beta > \gamma$; therefore, $\sin(180° - \beta) > \sin\gamma$. But, $\sin(180° - \beta) = \sin\beta$, hence again $\sin\beta > \sin\gamma$. But the sides $\mathbf{b}$ and $\mathbf{c}$ are in proportion to $\sin\beta$ and $\sin\gamma$, hence $\mathbf{b} > \mathbf{c}$.

15. Given $\alpha = 45°$, $\mathbf{b} = 17$, $\mathbf{c} = 17$.

Show that when two sides of a triangle are equal, the opposite angles are equal.

16. Show that when two sides of a triangle are equal, the line which divides their included angle into two equal parts is perpendicular to the third side and divides it equally.

17. Given $\alpha = 73°\ 12'$, $\mathbf{a} = 12$, $\mathbf{b} = 87$.

18. Given $\mathbf{a} = 13$, and all the angles equal to each other.

19. Given $\mathbf{a} = 43$, $\mathbf{b} = 68$, $\mathbf{c} = 25$.

Show that the half sum of the sides of a triangle must always be greater than any side, and so prove that the sum of any two sides is greater than the third, and their difference less than the third.

20. A tower 150 feet high, standing on a horizontal plane, casts a shadow 75 feet long. Find the sun's altitude.

21. A tower stands by a river. A person on the opposite bank finds the elevation of the top to be 60°; receding 40 yards in a direct line from the tower he finds the elevation 50°. What is the breadth of the river?

22. Wishing to know the distance between two inaccessible objects, A and B, I find a level place from which both can be seen, and measure then a distance $\overline{CD}$ equal to 134 feet. C is N. 5° E. from D; A is N. 45° 17′ E. from D, and B is N. 70° 41′ E. from D; B is due east from C, and A is N. 80° E. from C. What is the distance and direction of B from A? (See Fig. 34.)

23. The planet Mercury is said to be at its *greatest elongation* when lines drawn from it to the sun and the earth are perpendicular to each other. On March 29, 1879, Mercury was at its greatest elongation, and, as seen from the earth, was at an angular distance of 18° 57′ from the sun, the earth at that time being at a linear distance of 92,700,000 miles from the sun. How far was Mercury from the earth?

THE CIRCLE.

142. A point moving in a plane, so as to retain always a constant distance from a fixed point, describes a *circle*. The fixed point is called the *centre*, and the constant distance the *radius*. Any portion of the path of the point is called an *arc*, and the rectilinear distance between the extremities of an arc is its *chord*. When two lines pass through the centre of a circle, the angle which they make with each other is said to be subtended by the arc which they intercept.

143. The circle is a curve of *uniform curvature*, so that any two arcs described with equal radii may be so placed as to coincide throughout the whole extent of the shorter arc. For when the centres coincide, and the extremities of the shorter arc lie on the longer one, intermediate points of the two arcs cannot fail to coincide without being at unequal distances from the centre.

144. In § 18, the angle between two lines is conceived to pass through all successive values from 0° to 180° by the rotation of one line upon a point of another. In this rotation, any point of the moving line (AC, Fig. 35) describes an arc of a circle, and it is obvious that this arc increases in the same ratio with the angle. Or,

Arcs of the same circle are to each other as their subtended angles.

145. To *square the circle* is to find a numerical expression for the area of the inclosed figure, and to *rectify* it is to find the length of the circular periphery, the radius in each case being known. These problems are of great celebrity in the history of mathematics, but it has been demonstrated that the precise values of neither area nor length can be found, though an approximation may be made to any required degree of exactness. The simplest method of approximation is by means of polygons whose vertices lie in the curve, hence called inscribed polygons.

Let the entire curve (Fig. 36) be divided into $\mathbf{n}$ equal parts, the chords of these $\mathbf{n}$ arcs will constitute the perimeter of a polygon of $\mathbf{n}$ sides. Let lines pass through the centre, and extend to

each of these points of division. Then any one of these lines will make with the line next to it an angle of $\frac{360^\circ}{\mathbf{n}}$. It is obvious that the greater the number $\mathbf{n}$ is, the more nearly will the peripheries and areas of the two figures coincide; although for any finite value of $\mathbf{n}$, however large, the polygon must have a smaller area than the circle. Now the polygon is made up of $\mathbf{n}$ triangles, in each of which two sides and the included angle are known. The length of one of these sides being equal to the radius, may be called $\mathbf{r}$, while the included angle is $\frac{360^\circ}{\mathbf{n}}$. The remaining angles may easily be found, for since they are equal to each other (§ 141, Ex. 15), and their sum is $180^\circ - \frac{360^\circ}{\mathbf{n}}$, each of them is equal to $90^\circ - \frac{180^\circ}{\mathbf{n}}$. Hence the remaining side may be found by the proportion,

$$\sin\left(90^\circ - \frac{180^\circ}{\mathbf{n}}\right) : \sin\frac{360^\circ}{\mathbf{n}} :: \mathbf{r} : \text{the third side (§ 136).}$$

Now, $$\sin\left(90^\circ - \frac{180^\circ}{\mathbf{n}}\right) = \cos\frac{180^\circ}{\mathbf{n}};$$

and $$\sin\frac{360^\circ}{\mathbf{n}} = 2\sin\frac{180^\circ}{\mathbf{n}}\cos\frac{180^\circ}{\mathbf{n}} \quad (\text{§ } 102);$$

and by substituting these values in the proportion, the value of the third side is easily found to be $2\,\mathbf{r}\sin\frac{180^\circ}{\mathbf{n}}$.

The area of the triangle (Formula III. § 133) is $\frac{1}{2}\,\mathbf{r}^2\sin\frac{360^\circ}{\mathbf{n}}$.

As there are $\mathbf{n}$ triangles, the perimeter and area of the whole polygon will be found by multiplying these results by $\mathbf{n}$;

whence $$\text{perimeter} = 2\,\mathbf{rn}\sin\frac{180^\circ}{\mathbf{n}},$$

and $$\text{area} = \tfrac{1}{2}\,\mathbf{r}^2\mathbf{n}\sin\frac{360^\circ}{\mathbf{n}}.$$

By these formulæ may be computed the perimeter and area of inscribed polygons, having any required number of sides. For instance, if $\mathbf{n} = 6$, perimeter $= 2\,\mathbf{r} \times 6 \sin 30^\circ$, or $2\,\mathbf{r} \times 3$, while the area $= \frac{1}{2}\,\mathbf{r}^2 \times 6\sin 60^\circ = \mathbf{r}^2 \times \frac{6}{4}\sqrt{3}$. If $\mathbf{n} = 12$, perimeter $= 2\,\mathbf{r} \times 12 \sin 15^\circ$, and area $= \frac{1}{2}\,\mathbf{r}^2 \times 12 \sin 30^\circ = 3\,\mathbf{r}^2$.

By the aid of the table of sines the following may be computed and indefinitely extended:

Name of Polygon.	Value of $\mathbf{n}$.	Perimeter $= 2\,\mathbf{r} \times$	Area $= \mathbf{r}^2 \times$
Triangle	3	2.59808	1.29904
Hexagon	6	3.00000	2.59808
Dodecagon.	12	3.10582	3.00000
	24	3.13263	3.10582
	48	3.13935	3.13263
	96	3.14104	3.13935
	192	3.14145	3.14104
	384	3.14156	3.14145
	768	3.14158	3.14156

Here the successive values of $\mathbf{n}$ are each double the preceding, but the values of the perimeter and area are far from increasing at the same rapid ratio. In fact their increase, rapid for the first two or three terms, becomes much less so as we proceed, until we find that by doubling the number of sides the perimeter is increased only in the ratio of 3.14158 : 3.14156 while $\mathbf{n}$ is no greater than 768. This is due to the principle (explained in § 110), that the sines of very small angles are nearly in the ratio of the angles themselves. Hence, if $\mathbf{n}$ and $\mathbf{m}$ are very large numbers, $\frac{180^\circ}{\mathbf{n}}$ and $\frac{180^\circ}{\mathbf{m}}$ being very small angles, we have the proportion,

$$\sin\frac{180^\circ}{\mathbf{n}} : \sin\frac{180^\circ}{\mathbf{m}} :: \frac{180^\circ}{\mathbf{n}} : \frac{180^\circ}{\mathbf{m}} \qquad \textit{nearly.}$$

But $$\frac{180^\circ}{\mathbf{n}} : \frac{180^\circ}{\mathbf{m}} :: \mathbf{m} : \mathbf{n}.$$

Hence $$\sin\frac{180^\circ}{\mathbf{n}} : \sin\frac{180^\circ}{\mathbf{m}} :: \mathbf{m} : \mathbf{n} \qquad \textit{nearly;}$$

or $$\mathbf{m}\sin\frac{180^\circ}{\mathbf{m}} = \mathbf{n}\sin\frac{180^\circ}{\mathbf{n}} \qquad \textit{nearly.}$$

Thus if $\mathbf{m} = \frac{1}{2}\mathbf{n}$, we have $\frac{1}{2}\mathbf{n}\sin\frac{180^\circ}{\frac{1}{2}\mathbf{n}}$, or $\frac{1}{2}\mathbf{n}\sin\frac{360^\circ}{\mathbf{n}}$, nearly

equal to $\mathbf{n} \sin \frac{180°}{\mathbf{n}}$, so that the formula for area, when $\mathbf{n}$ is very great, may be written $\mathbf{r}^2\mathbf{n} \sin \frac{180°}{\mathbf{n}}$. Or, without restricting the value of $\mathbf{m}$ to $\frac{1}{2}$ $\mathbf{n}$, it may be said that the formula $\mathbf{n} \sin \frac{180°}{\mathbf{n}}$, when $\mathbf{n}$ is increased, rapidly approaches a fixed value, and may be brought within any required closeness of approximation to this value by sufficiently increasing $\mathbf{n}$. This fixed quantity, the value of $\mathbf{n} \sin \frac{180°}{\mathbf{n}}$, when $\mathbf{n} = \infty$, is denoted by the symbol π, and as the polygon is brought into coincidence with the circle by the indefinite increase of $\mathbf{n}$, we have

Periphery of circle

$$\left(= 2\,\mathbf{rn} \sin \frac{180°}{\mathbf{n}} \;[\text{when } \mathbf{n} = \infty]\right) = 2\,\pi\mathbf{r},$$

Area of circle

$$\left(= \tfrac{1}{2}\,\mathbf{r}^2\mathbf{n} \sin \frac{360°}{\mathbf{n}} = \mathbf{r}^2\mathbf{n} \sin \frac{180°}{\mathbf{n}} \;[\text{when } \mathbf{n} = \infty]\right) = \pi\mathbf{r}^2.$$

146. The numerical value of π may be found to six decimal places by giving $\mathbf{n}$ in the formula $\mathbf{n} \sin \frac{180°}{\mathbf{n}}$ a value no greater than 10800. $\frac{180°}{\mathbf{n}}$ then becomes $1'$, and (§ 110)

$$10800 \sin 1' = 10800 \times .0002908882 = 3.1415926.$$

The value in common use, when great accuracy is not required, is 3.1416, and frequently the value $\frac{22}{7}$ is found to be exact enough in practice. The computation of π has, however, been carried to hundreds of decimal places. The value to fifty places is

3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510.

147. The arc which is equal in length to the radius subtends an angle of $\frac{180°}{\pi}$, or $57°\ 17'\ 44''.8$.

For since an arc whose length is $\pi\mathbf{r}$, or half the periphery, subtends an angle of 180°, the arc whose length is $\mathbf{r}$ subtends an angle of $\frac{180°}{\pi}$ (§ 144).

148. Another approximate computation of the circular area, independent of the foregoing, may be made by the method of §§ 114, 121. Let rectangular axes pass through the centre of the curve (Fig. 37); they will divide the figure into four equal parts. The distance on the axis of abscissas over which the generatrix must move to describe one of these quarters is equal to the radius. Let this distance be divided into any number, say twenty, of equal parts or units, then the ordinate at any point of division may be found by subtracting the square of the abscissa of the point from the square of twenty, and extracting the square root of the remainder (§ 47). Finding in this way the length of every second ordinate, ten results are obtained:

$\sqrt{399}$, $\sqrt{391}$, $\sqrt{375}$, $\sqrt{351}$, $\sqrt{319}$, $\sqrt{279}$, $\sqrt{231}$, $\sqrt{175}$, $\sqrt{111}$, and $\sqrt{39}$; or

20.0, 19.8, 19.4, 18.7, 17.9, 16.7, 15.2, 13.2, 10.5, and 6.2.

The sum is 157.6, and the mean of these ten ordinates 15.76, or 0.788 of the radius. Taking this as the mean ordinate, the area is found by multiplying the ordinate by the radius (on the axis of abscissas), and the product by four. If the true value of the mean ordinate be computed from the formula $A = \pi \mathbf{r}^2$, using the value of π found in § 146, it will be found that it differs from the above mean of ten ordinates by only 0.003 of the radius.

149. If two circles are described with different radii, their peripheries are as the radii, and their areas as the squares of the radii.

For let the radii be $\mathbf{r}$ and $\mathbf{r}'$, then the peripheries are $2\pi\mathbf{r}$ and $2\pi\mathbf{r}'$, and the areas $\pi\mathbf{r}^2$ and $\pi\mathbf{r}'^2$.

But $\quad 2\pi\mathbf{r} : 2\pi\mathbf{r}' :: \mathbf{r} : \mathbf{r}'$,

and $\quad \pi\mathbf{r}^2 : \pi\mathbf{r}'^2 :: \mathbf{r}^2 : \mathbf{r}'^2$.

Also,

Two arcs subtending equal angles are as the radii with which they are described, since they are proportional parts of their respective circles.

Examples.

1. If the equator of the earth is a circle, whose diameter is 7925.6 miles, what is its circumference?

2. If the meridian passing through Colorado Springs were a circle of the same diameter, what would be the perpendicular distance of that place from the axis of the earth, its latitude being 38° 51′?

3. The longitude of Colorado Springs is 23° 18′ W. from Washing-

ton, while the latitude of the two places is nearly equal: Required, the distance which would be traversed by a person journeying due east from Colorado Springs to Washington.

4. The planet Saturn is surrounded by several flat rings, the outermost of which is estimated to have a circumference of 527,000 miles, with a breadth of 10,000 miles: Required, the area of this ring; also its distance from the surface of the planet, the diameter of the latter being 70,500 miles.

5. A circular pond has an area of a quarter of an acre; find the area of a driveway fifteen feet wide around the pond.

150. The ratio of the arc to the radius is a new function of the subtended angle, comparable with the sine, cosine, etc. This function of any angle a may be denoted by the expression *arc* a,— an expression which will represent the *length* of the arc only when $\mathbf{r} = 1$. The arc differs from all the other functions in this important respect, that the arcs of two angles are in the same ratio as the angles themselves. (See Fig. 38, where the ratio $\frac{\overline{BP}}{\overline{AB}} = \frac{\overline{CM}}{\overline{DM}} = \frac{\overline{DN}}{\overline{AN}} = \frac{\overline{ER}}{\overline{AE}}$, the numerators being *arcs*, not rectilinear distances. This constant ratio of arc to radius is the function *arc* a.)

151. The ratio of the two functions, sine and arc, of a given angle is sometimes required. If $\mathbf{n}$ denote the ratio of the given angle a to 180°,

then $\frac{\sin a}{\operatorname{arc} a}$ may be written $\frac{\sin \frac{180^\circ}{\mathbf{n}}}{\operatorname{arc} \frac{180^\circ}{\mathbf{n}}}$.

If both numerator and denominator be multiplied by $\mathbf{r}$,

this fraction becomes $\frac{\mathbf{r} \sin \frac{180^\circ}{\mathbf{n}}}{\mathbf{r} \operatorname{arc} \frac{180^\circ}{\mathbf{n}}}$,

where the denominator is the *length of the arc*. Now if each term be again multplied by $\mathbf{n}$ the denominator becomes the length of that arc which subtends 180°, that is. $\pi\mathbf{r}$. Hence the fraction may be written

$$\frac{\mathbf{rn} \sin \frac{180^\circ}{\mathbf{n}}}{\pi\mathbf{r}}, \quad \text{or} \quad \frac{\mathbf{n} \sin \frac{180^\circ}{\mathbf{n}}}{\pi}.$$

In this form the value of the ratio $\frac{\sin a}{\text{arc}\, a}$ may be found for any angle a by reference to a table of sines.

Thus, if $a = 90°$, $\mathbf{n} = 2$, and as $\sin 90° = 1$, we have,

$$\frac{\sin 90°}{\text{arc } 90°} = \frac{\mathbf{n} \sin \frac{180°}{\mathbf{n}}}{\pi} = \frac{2 \times 1}{\pi} = \frac{2}{\pi} = .63662.$$

If $\quad a\ 57°\ 17'\ 44''.8, \quad \mathbf{n} = \pi \quad$ and $\frac{\sin a}{\text{arc } a} = \sin 57°\ 17'\ 44''.8 = .84147.$

If $\quad a = 30°, \quad \mathbf{n} = 6 \quad$ and $\frac{\sin a}{\text{arc } a} = \frac{6 \times \frac{1}{2}}{\pi}\ \frac{3}{\pi} = .95493.$

When $\quad a = 0°, \mathbf{n} = \infty$, and the numerator, $\mathbf{n} \frac{\sin 180°}{\mathbf{n}}$ becomes equal to π (§ 145), hence the fraction is $\frac{\pi}{\pi}$ or 1. That is: the ratio $\frac{\sin a}{\text{arc } a}$ when $a = 0$, is equal to unity.

152. In Fig. 39 the relation of the circle to all the functions of an angle is shown. The angle a is represented in four different values, — one between 0° and 90°, one between 90° and 180°, one between 180° and 270°, and one between 270° and 360°. Rectangular axes pass through the centre of the circle, and of these the axis of abscissas may, for the present purpose, be called simply *the axis*. The point O, in which the axis intersects the curve at the right, is called the *origin of axes;* and the point Q, at the intersection of the axis of ordinates with the upper portion of the curve, is the *secondary origin*. The point B, C, D, or E, is called the *extremity* of the arc, and the line AB, AC, AD, or AE, the *bounding line*. The radius is denoted by the letter R.

Angle a is the angle made with the axis by the bounding line.

$\mathbf{r}$ arc a = the arc OB, OC, OD, or OE, measured from the origin of arcs around to the extremity of the arc in a direction contrary to the motion of the hands of a watch. (See § 60.)

$\mathbf{r} \sin a = \overline{FB}, \overline{GC}, \overline{HD}$, or $\overline{IE}$, the ordinate of the extremity of the arc or the distance from the axis to the extremity of the arc, in a direction perpendicular to the axis. When $\sin a$ is positive, i.e., when a is between 0° and 180°, this distance will be measured upward, but when $\sin a$ is negative, or a between 180° and 360°, it will be measured downward.

$\mathbf{r} \cos a = \overline{KB}, \overline{LC}, \overline{MD}$, or $\overline{NE}$, the abscissa of the extremity

of the area, or the distance from the axis of ordinates to the extremity, in a direction parallel to the axis AX. This distance is measured to the right when cos a is positive, i.e., when a is between 0° and 90°, or between 270° and 360°, and to the left when a is between 90° and 270°.

r tan $a = \overline{OP}$, $\overline{OR}$, $\overline{OP}$, or $\overline{OR}$, the distance from the origin of arcs to the bounding line in a direction perpendicular to the axis. This distance is measured upward when tan a is positive, i.e., when a is between 0° and 90°, or between 180° and 270°, but downward when tan a is negative.

r cot $a = \overline{QV}$, $\overline{QZ}$, $\overline{QV}$, or $\overline{QZ}$, the distance from the secondary origin to the bounding line in a direction parallel to the axis, measured to the right when the value of a is such as to make cot a positive, but to the left when cot a is negative.

The reciprocals of the cosine and sine of an angle are called the *secant* and *cosecant* repectively (abbreviated *sec* and *cosec*). The secant of an angle has always the same sign as the cosine, and the cosecant the same as the sine.

r sec $a = \overline{AP}$, $\overline{AR}$, $\overline{AP}$, or $\overline{AR}$, the distance from the centre, measured on the bounding line, to the point where that line meets a line perpendicular to the axis and passing through the origin of axes. This distance is measured toward the extremity of the arc (the direction of the positive radius) when sec a is positive, but from the extremity when sec a is negative. That either of the above distances, $\overline{AP}$ for example, is equal to **r** sec a, may be shown as follows: —

$$\overline{AP}^2 = \overline{AO}^2 + \overline{OP}^2 \ (\S\ 47) = \mathbf{r}^2 + \mathbf{r}^2 \tan^2 a = \mathbf{r}^2 (1 + \tan^2 a)$$

$$= \mathbf{r}^2 \left(1 + \frac{\sin^2 a}{\cos^2 a}\right) (\S\ 49) = \mathbf{r}^2 \left(\frac{\cos^2 a + \sin^2 a}{\cos^2 a}\right)$$

$$= \mathbf{r}^2 \cdot \frac{1}{\cos^2 a} \quad (\S\ 48).$$

$$\text{Hence } \overline{AP} = \mathbf{r} \cdot \frac{1}{\cos a} = \mathbf{r} \sec a.$$

In the same way prove the following: —

r cosec $a = \overline{AV}$, $\overline{AZ}$, $\overline{AV}$, or $\overline{AZ}$, the distance from the centre, measured on the bounding line, to the point where the line meets a line parallel to the axis and passing through the secondary origin.

This distance is measured toward the extremity of the arc when cosec α is positive, but from the extremity when cosec α is negative.

MISCELLANEOUS EXAMPLES.

1. A ladder, whose foot remains in a given position, just reaches a window on one side of a street, and when turned about its foot just reaches a window on the other side. If the two positions of the ladder are at right angles to each other, and the heights of the windows 36 and 27 feet respectively, find the width of the street and the length of the ladder.

2. From the top of a hill are observed two consecutive milestones on a straight horizontal road running from the base of the hill. The angles of depression are found to be 45° and 30°. Required the height of the hill.

3. Prove that $\tan\beta + \cot\beta = \dfrac{1}{\sin\beta\cos\beta}$.

4. Show that $(\cos^2\theta - 1)(\cot^2\theta + 1) = -1$.

5. Show that $\tan^2\gamma - \tan^2\beta = \dfrac{\cos^2\beta - \cos^2\gamma}{\cos^2\beta\cos^2\gamma}$

6. What angles have cosines equal to their cotangents?

7. Show that $\tan\gamma + \tan\beta = \dfrac{\sin(\gamma+\beta)}{\cos\gamma\cos\beta}$.

8. Show that $\cot(\theta + 45°) = \dfrac{\cot\theta - 1}{\cot\theta + 1}$.

9. Show that $\dfrac{\cos 27° - \sin 27°}{\cos 27° + \sin 27°} = \tan 18°$.

10. Show that $\cot\theta - 2\cot 2\theta = \tan\theta$.

11. If the angle γ of a triangle be 120° show that $\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{ab} + \mathbf{b}^2$.

12. The sides of a triangle are in arithmetical progression, and one angle is 90°. Required the other angles.

13. The sides of a triangle are in geometrical progression, and one angle is 90°. Required the other angles.

14. Two buoys, B and C, in a harbor, are 500 yards apart, while a third, A, is 1,000 yards from B and 700 yards from C. A boatman desires to know the position of his boat, from which he sees C in a line directly behind A, while the line to B makes with the former an angle of 18°.

15. An observer at Colorado Springs notices the sun going down just behind the top of Pike's Peak, at a time when its altitude above the horizon is 8° 47′. Required the elevation of the mountain above the level of the town, assuming that the horizontal distance is ten miles.

16. The length $\overline{AB}$ of a rectangular sheet of paper is **a** inches, while its breadth, $\overline{AC}$, is **b** inches. The paper is folded so that vertex A lies on the side $\overline{CD}$, the crease passing through the vertex B. Required the area of the part folded down.

GEOMETRY OF THREE DIMENSIONS.

151. In the foregoing §§, since § 17, the lines, angles, etc., which have been considered have been limited to a single plane. This restriction is now to be removed, and the remaining §§ will treat of the relations of lines situated in different planes.

In all applications, therefore, of preceding propositions to the reasoning of the following §§, the restrictions under which those propositions were demonstrated must be remembered, and no wider signification must be allowed than that originally intended. Thus, the proposition of § 31 means no more than this: "Two lines are parallel if they are perpendicular to the same line, *and lie in the same plane with it and with each other.*"

152. But the theorem of § 26, though introduced after this restriction was imposed, does not depend upon it for demonstration. Whenever two or more lines which are parallel to each other are intersected by other lines also parallel to each other, the angles at any one intersection are equal to those at any other, even though the plane of the two lines which meet in one intersection contains neither of the lines which meet at the other. For

the difference between two directions must remain unchanged as long as the directions themselves remain so.

153. If from any point of a plane a line be drawn parallel to a given line of the plane, it will lie wholly in the plane.

For a line may be drawn *in the plane*, parallel to the given line, through the given point; and since two different lines cannot extend from the same point in the same direction, *no other* parallel to the given line can be drawn through the given point.

154. As a line is designated by naming two points upon it, so the position of a plane is sufficiently determined by naming two lines as contained in it, which either intersect or are parallel. For if they intersect, one may be used as the generatrix and the other as the directrix of the plane; and if they are parallel, either may be the directrix, while the generatrix is any line joining a point of one of them with a point of the other (§§ 14, 15).

155. The system of Cartesian coördinates with which the student has become familiar is rendered applicable to points, etc., not lying in one plane, by the use of three converging axes instead of two, whence the name "Geometry of Three Dimensions." If AF, AG (Fig. 40), be the axes of coördinates for the plane in which they are situated, then AH may be the third axis, provided it does not lie in the plane of the other two.

156. Conceive right lines to extend in the plane of AF and AG, through the point A, in an indefinite number of directions. AH cannot coincide with any of these, else it would also lie in the plane of AF and AG. Hence it must make a finite angle with each of them. In other words, of the angles made by AH with the various lines which it meets in the plane there is one angle greater than zero, which is the *minimum angle*, or such that *no less angle* can be made with AH by any line of the plane. This angle, whatever be its value, is called the *inclination* of the line AH to the plane of AF and AG, and will be denoted by the letter ι. Let AM (in the figure) be a line of the plane of AF and AG, which makes with AH the angle ι.

157. In the same figure, the line AM′ which extends in the direction opposite to AM in the plane of AF and AG, must make with AH an angle of $180°-\iota$. So, if AS be any other line of the plane

of AF and AG, which passes through A and makes with AH an angle $\varkappa$, the angle of AS′ with AH is $180°—\varkappa$. But as $\varkappa$ was not less than ι, therefore $180°—\varkappa$ is not greater than $180—\iota$; in other words, AM′ makes with AH a *maximum* angle. If a line passing through A be conceived to be rotated about this point, while remaining in the plane of AF and AG, it will change its direction by insensible variations, hence its angle with AH, changing in the same manner, will pass through all possible values between ι and $180°—\iota$.

In such a range of values, the value 90° cannot fail to be included, whatever the angle ι may be; hence, whenever a line pierces a plane, some line of that plane is perpendicular to the first-mentioned line.

158. Of all the distances measured on right lines from various points of a plane to any one point of a given line which pierces that plane, no one is shorter than that which is measured upon a line perpendicular to that line of the plane with which the given line makes its minimum angle.

That is, in Fig. 41, if A′M be that line of the plane of AF and AG with which AH makes its minimum angle ι, P any point of AH, and RP a perpendicular to AM through P, then no rectilinear distance $\overline{SP}$ between the plane and the point P can be shorter than RP.

Let $\overline{QP}$ be the perpendicular distance of P from the line AS, let $\varkappa$ be the less of the two supplementary angles which AH makes with AS, and let μ be the angle between SQ and SP.

Then $\overline{QP} = \overline{AP} \sin \varkappa$ and $\overline{QP} = \overline{SP} \sin \mu$,

whence $\overline{SP} = \overline{AP} \dfrac{\sin \varkappa}{\sin \mu}$. Also, $\overline{RP} = \overline{AP} \sin \iota$.

Hence $$\overline{SP} : \overline{RP} :: \frac{\sin \varkappa}{\sin \mu} : \sin \iota.$$

Now, as neither $\varkappa$ nor ι exceeds 90°, and $\varkappa$ is not less than ι, $\sin \varkappa$ cannot be less than $\sin \iota$, and as $\sin \mu$ cannot exceed 1, $\dfrac{\sin \varkappa}{\sin \mu}$ cannot be less than $\sin \varkappa$; hence $\dfrac{\sin \varkappa}{\sin \mu}$ is not less than $\sin \iota$, and $\overline{SP}$ is not less than $\overline{RP}$.

159. When a line pierces a plane, that line of the plane which makes with the former a minimum angle, also makes a minimum angle with any other line which lies in the plane of the two former lines.

Thus, in Fig. 42, no line of the plane of AF and AG can make a less angle than AM makes with any line (BK or EC) of the plane of AH and AM.

In the case of BK, a line parallel to AH, the proposition is evident. For every line of the plane of AF and AG that passes through B is parallel to a corresponding line passing through A, and hence makes with BK an angle equal to that which its parallel makes with AH (§ 152). Hence, as no line makes with AH a less angle than AM makes, none will make a less angle with BK.

But to prove the proposition for EC, a line extending from any point E of AM in a direction not parallel to AH, conceive a line parallel to AH to pass through C and meet AM in D. Also, let a line pass through C perpendicular to AM and meeting it in L. Then no line of the plane of AF and AG makes a less angle with DC than that which AM makes. Hence no rectilinear distance shorter than $\overline{LC}$ connects C with the plane of AF and AG (§ 158).

But if any line of that plane, passing through E, made with EC a less angle than AM does, then the perpendicular distance from that line to C would be shorter than $\overline{LC}$. As this has been shown to be impossible, it follows that no line of the plane of AF and AG can make with EC a less angle than AM makes with it.

160. If through any line which pierces a plane, and through that line of the plane which makes with the former a minimum angle, another plane pass, so that the second-named line is common to the two planes, then any line drawn in either plane perpendicular to this common line, is perpendicular to every line which it meets in the other plane.

In Fig. 43, if AM be, as heretofore, that line of the plane of AF and AG which makes with AH the minimum angle, then AO drawn in the plane of AH and AM perpendicular to AM, is perpendicular to every line in the plane of AF and AG which passes through A.

For no such line can make with it an angle less than its angle with AM (§ 159); that is, less than 90°. Also (§ 157), no such line can make with it an angle greater than 180°—90°, i. e., than 90°. Hence all lines of this plane which meet AO in A make with it angles neither less nor greater than 90°, or it is perpendicular to them all.

On the other hand, any line as AN of the plane of AF and AG drawn perpendicular to AM is perpendicular to every line of the plane of AH and AM which meets it in A. For, let a line be drawn connecting any point U of this plane with any point V of the line AN. Let there be drawn, moreover, a line UW perpendicular to AM; also a line joining U and V, and a line joining V and W. It has been already shown, in this §, that UW will be perpendicular to VW.

Now $\overline{UV}^2 = \overline{VW}^2 + \overline{WU}^2$ but $\overline{VW}^2 = \overline{VA}^2 + \overline{AW}^2$

hence $\overline{UV}^2 = \overline{VA}^2 + \overline{AW}^2 + \overline{WU}^2$.

Also $\overline{AU}^2 = \overline{AW}^2 + \overline{WU}^2$ hence $\overline{UV}^2 = \overline{VA}^2 + \overline{AU}^2$.

This equation indicates that AU is perpendicular to AV or AN, for if we call the angle of these two lines α and the distance $\overline{UV}$, $\mathbf{a}$ and put $\mathbf{b}$ for $\overline{VA}$ and $\mathbf{c}$ for $\overline{AU}$, we have —

$$\alpha^2 = \beta^2 + \gamma^2.$$

But (Equation (15) § 139)

$$\cos \alpha = \frac{\mathbf{b}^2 + \mathbf{c}^2 - \mathbf{a}^2}{2\,\mathbf{bc}}.$$

Hence $\cos \alpha = 0$, and therefore $\alpha = 90°$, or AN is perpendicular to AU.

161. When a line which pierces a plane is perpendicular to every line which it meets in the plane, it is said to be *perpendicular to the plane.*

Thus AO is perpendicular to the plane of AF and AG, and AN is perpendicular to the plane of AH and AM.

The *foot* of the perpendicular is the point in which it pierces the plane.

162. If a line is perpendicular to a plane, every line parallel to the former is also perpendicular to the same plane.

For every line drawn in the plane through the point in which it

is pierced by the second line makes right angles with that line, because it is parallel to some line of the same plane which makes right angles with the first line.

163. Through a given point there can pass only one line perpendicular to a given plane.

For if the point be without the plane, and two perpendiculars be drawn from it to the plane, the line joining their feet is at right angles to both, while the three lines are in one plane (§§ 154, 161), which is impossible (§ 31). But if the given point be upon the plane, and two perpendiculars be drawn from it, then from some other point of one of them let a line pass parallel to the other; it will be perpendicular to the plane (§ 162), which is impossible as before.

164. The inclination of a line to a plane is the complement of the angle which the line makes with a perpendicular to the plane.

In Fig. 41, if a line PR be drawn from any point P of AH perpendicular to AM, it is (§ 160) the perpendicular from that point to the plane of AF and AG. But the three lines AP, PR, and AR are in one plane, hence (§ 30) ι is the complement of the angle between PR and PA.

165. Hence, two lines perpendicular to the same plane are parallel. For the angle between them is the complement of 90°, the inclination of either to the plane, and is therefore 0°.

166. If a line be perpendicular to a plane, any line perpendicular to the first line, and passing through its foot, lies in the mentioned plane.

Let AT (Fig. 44) be perpendicular to AO at A, while AO is perpendicular to the plane of AF and AG, and if AT do not lie in that plane, let AM be that line of the plane which makes with AT the minimum angle; and through T let a line JT pass perpendicular to AM, then (§ 160) it will also be perpendicular to the plane of AF and AG, hence (§ 165) parallel to AO, and therefore perpendicular to AT because AO is. Hence we have two perpendiculars, AJ and AT, from the same point A upon the same line, and the three lines in one plane, which is impossible.

167. If one of two lines remain fixed in position, while the other is rotated about the former, maintaining with it a constant angle of 90°, the second line describes a plane perpendicular to the first.

For, as shown in the last §, it can assume no position not lying in a plane perpendicular to the first line through the point of intersection; and in rotating entirely around that line it will evidently pass over the whole of such a plane.

168. The construction of the system of rectangular Cartesian coördinates for the geometry of three dimensions can now be explained. (See Fig. 45.) From the origin A of rectangular coördinates in the plane of AX and AY, a third axis AZ is extended in a direction perpendicular to that plane. Now, let a plane perpendicular to AX be formed by the rotation of AY (§ 167), then AZ will lie in that plane (§ 166). In the same way, let AX revolve about AY, a plane perpendicular to AY will be formed, which will also contain AZ. Finally, the plane of AX and AY may itself be considered as generated by the revolution of either of those lines about AZ. It thus appears that there are not only three coördinate axes, but three coördinate planes, each axis being a line common to two of the planes, and perpendicular to the third. The three coördinates of any point are its distances, parallel to each of the three axes respectively, from the plane of the other two; thus, in the figure, the coördinates of a point P are $\overline{QP}$, $\overline{RP}$, and $\overline{SP}$. If through Q, R, and S, the points in the coördinate planes from which these coördinates extend, lines are drawn in these planes parallel to the axes (and hence parallel also to QP, RP, and SP), then any two of the three lines QP, RP, and SP will be in the same plane with two of the lines so drawn. Thus QP and RP are in the same plane with LR and LQ (§ 154), which lines therefore meet the axis AZ in one point L.

Hence $\overline{LR} = \overline{QP}$ and $\overline{LQ} = \overline{RP}$. (§ 35.)

In like manner

$$\overline{MS} = \overline{QP} \quad \text{and} \quad \overline{MQ} = \overline{SP};$$

Also $\overline{NS} = \overline{RP}$ and $\overline{NR} = \overline{SP}$.

For a similar reason

$$\overline{AN} = \overline{MS} = \overline{LR}, \text{ and } \overline{AM} = \overline{NS} = \overline{LQ}, \text{ and } \overline{AL} = \overline{NR} = \overline{MQ}.$$

Therefore

$\overline{AN} = \overline{QP}$, and $\overline{AM} = \overline{RP}$, and $\overline{AL} = \overline{SP}$.

Hence the coördinates of a point, as in plane geometry, may be measured at pleasure upon the axes, as well as parallel to them.

THE MENSURATION OF VOLUMES.

169. A volume has been defined (§ 17) as generated by the motion of a plane figure along a line not lying in its plane. In its motion the plane figure is assumed to retain a constant inclination to the directrix upon which it moves. Its form and area may remain unchanged, or either or both may change continuously in accordance with some fixed law. To determine the volume generated in the latter case, it is necessary to know the mean area of the moving figure. If a number of points be taken at equal intervals upon the directrix, and the area of the figure at each of these points determined, the sum of these areas, divided by the number of the points, is the mean of the area for those points, and if an expression for such a mean of areas can be put into a form which will still be applicable, when the number of points is extended to include every point of the directrix over which the generating surface moves, then the mean area of the surface has been found. It is assumed as self-evident, in accordance with the analogy of areas described by variable ordinates (§ 117), that the volume generated by a variable area, moving to a given distance upon a fixed line, is equal to that generated by a constant figure of the same plane, moving to the same distance upon the same line, provided that the constant area of the latter figure is equal to the mean area of the former.

170. As in the generation of areas by moving ordinates, so in the generation of volumes by moving figures, there are three quantities whose value affects that of the generated volume: 1st, the area of the generating figure; 2d, the distance on the directrix over which it moves, and 3d, the sine of the inclination of the directrix to the plane of the generatrix. If in the generation of a particular volume, any two of these quantities have values equal to the values of the same quantities in the case of another volume, while the third quantity has different values in the two cases, then the volumes generated will be as the values of that third quantity. In particular: —

171. *First.* — The volumes generated by different figures, parts of the same plane surface, the whole of which moves to a given distance along a fixed right line, will be as the areas of the figures.

Thus, in Fig. 46, if the figure M move to the distance $\overline{AB}$, or **m**, upon the axis of X, then the volume I described by the whole figure M is to the volume II, described by N, as M to N. This volume II described by N moving to the distance m (see the figure), may be also considered as generated by the figure P moving along the line AZ to the distance $\overline{AC}$ or **p**. Its value will then be to that of III, described by the figure Q in the latter motion, as P to Q. In like manner, volume **i** is to I as V to M:

$$\mathbf{ii} : \mathrm{II} :: \mathrm{W} : \mathrm{P}.$$

172. *Second.* — The volumes generated by the same figure, moving to different distances upon the same right line, are as the distances traversed.

Thus, if the figure N be stopped after passing over the distance $\overline{AD}$, or **n**, the resulting volume IV will be to II as **n** to **m**.

Hence $$\mathrm{IV} : \mathrm{I} :: \mathrm{N}\mathbf{n} : \mathrm{M}\mathbf{m}.$$

173. *Third.* — The volume generated by a given figure moving to a given distance varies directly as the sine of the inclination of the directrix upon which it moves to the plane of the figure.

Thus, in Fig. 47, as AZ is perpendicular to the plane of the figure P, the angle ι, made by the directrix AZ′ with AY (which is the complement of the angle it makes with AZ), is its inclination to the plane of the figure (§ 154). Now, if the distances $\overline{AL}$, $\overline{AM}$, and $\overline{AP}$ be called **l**, **m**, **p**, respectively, then the area of the figure N is **lp** sin ι (§ 121), while that of P is **lm**. Now let F denote the volume generated by a unit's area of the plane of the figure N moving to a unit's distance along the axis AX, which is perpendicular to it, and G the volume described by a unit's area of the plane of P moving to a unit's distance along the axis AZ′, inclined to it at the angle ι. Then from the two preceding §§ we have —

$$\text{Volume VI} : \mathrm{F} :: \mathbf{m}.\,\mathbf{lp} \sin \iota : 1,$$

and $$\text{Volume VI} : \mathrm{G} :: \mathbf{p}.\,\mathbf{lm} : 1,$$

whence $$\mathrm{G} : \mathrm{F} :: \mathbf{lmp} \sin \iota : \mathbf{lmp},$$

or $$\mathrm{G} : \mathrm{F} :: \sin \iota : 1.$$

So, if H were to represent the volume generated by a unit of area moving to a unit's distance along a directrix inclined to its plane at an angle ι', it might be shown in the same way that —

$$H : F :: \sin \iota' : 1,$$

whence $$G : H :: \sin \iota : \sin \iota';$$

that is, the volumes described by equal areas, moving to equal distances, are as the sines of the inclinations of the directrices to the planes of the generating figures.

174. When the directrix is perpendicular to the plane of the generating figure, the sine of its inclination is unity; hence —

In the measurement of volume the unit assumed is the volume generated by a plane figure of a unit's area, moving to a unit's distance along a line perpendicular to its plane.

175. The content of any volume is its ratio to the unit of the volume, or the number of times it contains that unit. Hence —

The content, or numerical measure of any volume is the continued product of the mean area of the generating figure, the distance on the directrix over which the latter moves, and the sine of the inclination of the directrix to the plane of the generating figure.

176. This formula will now be applied to the simplest case of volumes generated by variable areas, — that, namely, in which, in the motion of the generating figure, each point of its periphery describes a right line, and all these right lines converge in a single point, called the *apex* of the volume.

In this case, any one of these converging lines may be taken as the directrix, or the directrix may be a perpendicular from the apex upon the plane of the generating figure. Such a perpendicular will be in one plane with any one of the lines converging at the apex (§ 154).

177. If a line be drawn from any point of the generating figure to the foot of the perpendicular, then in any two positions of the generating figure this line will have the same direction. (Thus, in Fig. 48, MN is parallel to **mn**.) For such a line must remain in the plane containing the perpendicular AM and the line described by the point N, and is also in each position perpendicular to AM

(§§ 161, 169). Hence the distance ($\overline{MN}$ or $\overline{\mathbf{mn}}$) of the point from the foot of the perpendicular retains a constant ratio to the distance ($\overline{AM}$ or $\overline{A\mathbf{m}}$) on the perpendicular from the apex to the generating plane. That is, —

$$\overline{MN} : \overline{\mathbf{mn}} :: \overline{AM} : \overline{A\mathbf{m}}. \qquad (\S\ 37.)$$

So also if S, **s**, be any other point of the generating figure —

$$\overline{MS} : \overline{\mathbf{ms}} :: \overline{AM} : \overline{A\mathbf{m}},$$

Whence $$MN : MS :: \overline{\mathbf{mn}} : \overline{\mathbf{ms}}.$$

That is, in the motion of the generating figure, the distances between its several points retain constant ratios to each other, or any one such position or *state* of the generating figure is a *similar figure* to any other state (§ 123).

178. From the proportion —

$$\overline{MN} : \overline{\mathbf{mn}} :: \overline{AM} : \overline{A\mathbf{m}},$$

it is apparent that the ratio of the homologous distances of any two states of the generating figure is the ratio of the perpendicular distances of their planes from the apex. Or, since the distances of the generating plane from the apex, measured upon the perpendicular, will be to the distance on any other convergent, as AN, in a constant ratio, equal to the cosine of the angle which such a convergent makes with the perpendicular, it appears that the ratio of homologous distances, for any two states of the generating figure, will be equal to the ratio of the distances of that figure, in its two positions, from the apex, measured on the directrix. The ratio of the area will be the square of this (§ 123). That is, —

The area of the generating figure maintains a constant ratio to the square of its distance from the apex, measured on the directrix.

179. To find the mean area of the generating figure, and thence the measure of the generated volume, let ι denote the inclination of the directrix to the plane of the generatrix, and **h** the distance to $\overline{\mathbf{n}N}$ (Fig. 48), upon the directrix over which the generating figure passes, and let us suppose that the whole distance $\overline{AN}$ from

the apex is to $\overline{\mathbf{n}\mathrm{N}}$ as 1 to $\mathbf{r}$, where $\mathbf{r}$ is a fraction less than 1. Then —

$$\overline{\mathrm{AN}} = \frac{\mathbf{h}}{\mathbf{r}}, \quad \text{and} \quad \overline{\mathrm{A}\mathbf{n}} = \overline{\mathrm{AN}} - \overline{\mathbf{n}\mathrm{N}} = \mathbf{h}\,\frac{1-\mathbf{r}}{\mathbf{r}^2}.$$

If A denote the area of the generating figure when at the point N, its area $\mathbf{a}$ at the point $\mathbf{n}$ may be found from the proportion —

$$\mathbf{a} : \mathrm{A} :: \mathbf{h}^2\,\frac{(1-\mathbf{r})^2}{\mathbf{r}^2} : \frac{\mathbf{h}^2}{\mathbf{r}^2};$$

whence $\mathbf{a} = \mathrm{A}\,(1-\mathbf{r})^2.$

Let us now suppose that the distance $\mathbf{h}$ is divided into some number $\mathbf{n}$ of equal parts, then the distance from N to the $\mathbf{m}^{\text{th}}$ point of division, counting toward A, will be $\frac{\mathbf{mh}}{\mathbf{n}}$, hence the distance of the same point from A is —

$$\frac{\mathbf{h}}{\mathbf{r}} - \frac{\mathbf{mh}}{\mathbf{h}}, \quad \text{or} \quad \frac{\mathbf{h}}{\mathbf{rn}}\,(\mathbf{n}-\mathbf{rm}).$$

The area of the generating figure at the $\mathbf{m}^{\text{th}}$ point of division will then be found from the proportion —

$$\textit{area} : \mathrm{A} :: \frac{\mathbf{h}^2}{\mathbf{r}^2\,\mathbf{n}^2}\,(\mathbf{n}-\mathbf{rm})^2 : \frac{\mathbf{h}^2}{\mathbf{r}^2};$$

whence the area sought $= \frac{\mathrm{A}}{\mathbf{n}^2}\,(\mathbf{n}-\mathbf{rm})^2.$

If now we put for $\mathbf{m}$ the numbers 1, 2, 3, 4, etc., successively, we find the area of the generating figure at the successive points of division to be: —

$$\frac{\mathrm{A}}{\mathbf{n}^2}\,(\mathbf{n}-\mathbf{r})^2, \quad \frac{\mathrm{A}}{\mathbf{n}^2}\,(\mathbf{n}-2\mathbf{r})^2, \quad \frac{\mathrm{A}}{\mathbf{n}^2}\,(\mathbf{n}-3\mathbf{r})^2, \text{ etc.}$$

The sum of all these areas will be —

$$\frac{\mathrm{A}}{\mathbf{n}^2}\Big\{(\mathbf{n}^2-2\,\mathbf{nr}+\mathbf{r}^2)+(\mathbf{n}^2-4\,\mathbf{rn}+4\,\mathbf{r}^2)+(\mathbf{n}^2-6\,\mathbf{rn}+9\,\mathbf{r}^2)$$
$$+ \text{etc.}\Big\};$$

or, collecting the terms so as to bring like powers of $\mathbf{n}$ together, the sum of the areas is —

$$\frac{A}{\mathbf{n}^2}\Big\{(\mathbf{n}^2+\mathbf{n}^2+\mathbf{n}^2+\text{etc.})-(2+4+6+\text{etc.})\,\mathbf{rn}+(1+4+9+\text{etc.})\,\mathbf{r}^2\Big\}$$

where the number of terms collected in each parenthesis is $\mathbf{n}$.

Now, it is evident that the sum of $\mathbf{n}^2+\mathbf{n}^2+\mathbf{n}^2+$ etc. to $\mathbf{n}$ terms is $\mathbf{n}\times\mathbf{n}^2$, or $\mathbf{n}^3$; and it is shown in Algebra * that the sum of the series 2, 4, 6, etc., to $\mathbf{n}$ terms is $\mathbf{n}^2+\mathbf{n}$; also, that the sum of the series 1, 4, 9, etc., to $\mathbf{n}$ terms is —

$$\tfrac{1}{6}\,(2\,\mathbf{n}^3+3\,\mathbf{n}^2+\mathbf{n}).$$

Hence the total sum of the $\mathbf{n}$ areas is —

$$\frac{A}{\mathbf{n}^2}\Big\{\mathbf{n}^3-(\mathbf{n}^2+\mathbf{n})\,\mathbf{rn}+\tfrac{1}{6}\,(2\,\mathbf{n}^3+3\,\mathbf{n}^2+\mathbf{n})\,\mathbf{r}^2\Big\};$$

---

* The first of the two series (2, 4, 6, etc.) is simply an arithmetical progression. If the student has not learned the method of summing the second series (the series of squares), he may convince himself by the following method that the formula for the sum, as given above, is correct.

First let him test the formula for a few successive values of $\mathbf{n}$, beginning with unity. Thus —

$\frac{1}{6}(2\,\mathbf{n}^3+3\,\mathbf{n}^2+\mathbf{n})$, when $\mathbf{n}=1$, is 1; when $\mathbf{n}=2$, is 5; when $\mathbf{n}=3$, is 14.

But $1+4=5$ and $1+4+9=14$,

hence the formula holds true up to three terms. If now the formula be true for the sum of the first $\mathbf{m}-1$ squares, it is true for one more, i.e., for the sum of the first $\mathbf{m}$ squares. For, substituting $\mathbf{m}-1$ for $\mathbf{n}$ in the formula, it becomes —

$$\tfrac{1}{6}\,[2\,(\mathbf{m}-1)^3+3\,(\mathbf{m}-1)^2+(\mathbf{m}-1)],$$

which reduces to $\frac{1}{6}\,(2\,\mathbf{m}^3-3\,\mathbf{m}^2+\mathbf{m})$.

Now if to this sum of $\mathbf{m}-1$ squares we add the $\mathbf{m}^{th}$, viz., $\mathbf{m}^2$, the result is

$$\tfrac{1}{6}\,(2\,\mathbf{m}^3+3\,\mathbf{m}^2+\mathbf{m}),$$

which is evidently what would have been obtained by putting $\mathbf{m}$ for $\mathbf{n}$ in the formula.

This proves that if the formula correctly represents the sum of any definite number of terms, it is true for one more term. If, therefore, it is true for three terms, it must be true for four; if for four, for five, etc.; whence it is plain that it must be true for any number of terms, however great.

The same method of proof may be applied to show that $\mathbf{n}^2+\mathbf{n}$ is the sum of the progression.

which, by collecting terms again, may be written in the form —

$$\frac{A}{\mathbf{n}^2}\left\{(1-\mathbf{r}+\tfrac{1}{3}\mathbf{r}^2)\,\mathbf{n}^3-(\mathbf{r}-\tfrac{1}{2}\mathbf{r}^2)\,\mathbf{n}^2+\tfrac{1}{6}\mathbf{r}^2\mathbf{n}\right\}.$$

The mean of these $\mathbf{n}$ areas will now be found by dividing their sum by $\mathbf{n}$, and hence is —

$$\frac{A}{\mathbf{n}^3}\left\{(1-\mathbf{r}+\tfrac{1}{3}\mathbf{r}^2)\,\mathbf{n}^3-(\mathbf{r}-\tfrac{1}{2}\mathbf{r}^2)\,\mathbf{n}^2+\tfrac{1}{6}\mathbf{r}^2\mathbf{n}\right\}, \text{ or}$$

$$A\left\{(1-\mathbf{r}+\tfrac{1}{3}\mathbf{r}^2)-\frac{1}{\mathbf{n}}(\mathbf{r}-\tfrac{1}{2}\mathbf{r}^2)+\frac{1}{6\mathbf{n}^2}\mathbf{r}^2\right\}.$$

This formula expresses the mean of the $\mathbf{n}$ areas for any value of $\mathbf{n}$, great or small; but if we suppose that some one of the points of division falls at every point of the distance $\mathbf{h}$, the value of $\mathbf{n}$ becomes infinitely great, and the co-efficients $\frac{1}{\mathbf{n}}$ and $\frac{1}{6\mathbf{n}^2}$ are each equal to zero; so that all the terms of the formula after the first parenthesis vanish, and the true mean value of the variable area is found to be —

$$A\,(1-\mathbf{r}+\tfrac{1}{3}\mathbf{r}^2).$$

Hence, by § 175, the volume generated is equal to —

$$A\,(1-\mathbf{r}+\tfrac{1}{3}\mathbf{r}^2)\,\mathbf{h}\sin\iota.$$

180. This result is usually expressed in a slightly modified form, obtained as follows: If H denote the *perpendicular* distance between the two positions of the generating figure at M and $\mathbf{m}$, and δ denote the angle between this perpendicular AM and the directrix AN, then $\mathbf{h}\cos\delta = H$. But since δ is the complement of ι (§ 164) $\cos\delta = \sin\iota$, whence $\mathbf{h}\sin\iota = H$. Also, the expression $A\,(1-\mathbf{r}+\frac{1}{3}\mathbf{r}^2)$ is equivalent to $\frac{1}{3}(3A - 3A\mathbf{r} + A\mathbf{r}^2)$, which may be written —

$$\tfrac{1}{3}(A+A)(1-\mathbf{r})+A\,(1-2\mathbf{r}+\mathbf{r}^2).$$

Now $\quad A\,(1-2\mathbf{r}+\mathbf{r}^2)=A\,(1-\mathbf{r})^2=\mathbf{a}, \qquad$ (§ 179)

and $\quad A\,(1-\mathbf{r})=\sqrt{A\mathbf{a}},$

the mean proportional between A and $\mathbf{a}$. Hence the expression for the volume may be written:

$$\tfrac{1}{3}H\,(A+\sqrt{A\mathbf{a}}+\mathbf{a}).$$

In words: —

When a volume is generated by a plane figure moving so that every point of the figure describes a right line, and all these right lines converge at one point, then the volume generated between any two positions of the generating figure, both of which are on the same side of the point of convergence, is equal to the product of the perpendicular distance between these positions, by one-third the sum of the areas of the generating figure in these two positions, and of a mean proportional between them.

181. A line joining two points of the generating figure retains the same direction in all positions of the figure, as may easily be shown from § 177; and hence in the motion of the generating figure such a line describes a plane (§ 12). Hence, when the generating figure is bounded by right lines, the surface of the generated volume is composed of plane figures (polygons), whose area may be found in any given case by the application of the formulæ for area already deduced (§§ 127, 128).

182. Besides the species of volume already considered in §§ 176–181, another variety is generated by a parallelogram moving along a directrix which passes through the intersection of its diagonals, and is inclined to its plane at an angle ι. The sides of the parallelogram retain their direction as they move, but vary in length, the difference between the lengths of a given side in two positions of the generating figure being in a fixed ratio to the distance on the directrix intervening between the two positions. From these conditions it may be shown that the four vertices of the generating figure move in right lines which intersect each other two and two, hence the surface of the volume consists of plane figures. It may be assumed, with sufficient generality for practical purposes, that no two of these lines intersect within the space passed over by the generating plane. To find the mean area of the generating parallelogram, let $\mathbf{p}$ and $\mathbf{q}$ be the lengths of two adjacent sides at the beginning of the motion, and let $\mathbf{p}\,(1+\mathbf{r})$, and $\mathbf{q}\,(1+\mathbf{r}')$ be their length respectively at the end of the motion. Then if λ denote the angle which they make with each

other, $\mathbf{pq} \sin \lambda$ is the area of the parallelogram when the motion begins, and $\mathbf{pq} \sin \lambda\,(1+\mathbf{r})\,(1+\mathbf{r}')$ is its final area. If the distance $\mathbf{h}$ on the directrix which is traversed by the motion be divided into $\mathbf{n}$ equal parts, then when the generating figure has reached the $\mathbf{m}^{\text{th}}$ point of division, its sides will be $\mathbf{p}\left(1+\frac{\mathbf{m}}{\mathbf{n}}\mathbf{r}\right)$ and $\mathbf{q}\left(1+\frac{\mathbf{m}}{\mathbf{n}}\mathbf{r}'\right)$, and its area $\mathbf{pq} \sin \lambda \left(1+\frac{\mathbf{m}}{\mathbf{n}}\mathbf{r}\right)\left(1+\frac{\mathbf{m}}{\mathbf{n}}\mathbf{r}'\right)$.

Putting for $\mathbf{m}$ the numbers 1, 2, 3, etc., successively, and expanding the parentheses, we find for the successive values of the area the following: —

$$\mathbf{pq} \sin \lambda \left[1+\frac{1}{\mathbf{n}}(\mathbf{r}+\mathbf{r}')+\frac{1}{\mathbf{n}^2}\,\mathbf{rr}'\right];$$

$$\mathbf{pq} \sin \lambda \left[1+\frac{2}{\mathbf{n}}(\mathbf{r}+\mathbf{r}')+\frac{4}{\mathbf{n}}\,\mathbf{rr}'\right];$$

$$\mathbf{pq} \sin \lambda \left[1\times\frac{3}{\mathbf{n}}(\mathbf{r}+\mathbf{r}')+\frac{9}{\mathbf{n}^2}\,\mathbf{rr}'\right];\ \text{etc.}$$

The sum of these n areas will be —

$$\mathbf{pq} \sin \lambda \Big[(1+1+1+\text{etc.})+(1+2+3+\text{etc.})\frac{\mathbf{r}+\mathbf{r}'}{\mathbf{n}}$$
$$+(1+4+9+\text{etc.})\frac{\mathbf{rr}'}{\mathbf{n}^2}\Big],$$

where each parenthesis contains $\mathbf{n}$ terms. The sum of $\mathbf{n}$ terms of the series $1+1+1+$ etc. is $\mathbf{n}$, of the series $1+2+3+$ etc. is $\frac{\mathbf{n}^2+\mathbf{n}}{2}$, and of the series $1+4+9+$ etc. is $\frac{2\,\mathbf{n}^3+3\,\mathbf{n}^2+\mathbf{n}}{6}$; hence the sum of the n areas is —

$$\mathbf{pq} \sin \lambda \Big[\mathbf{n}+\frac{1}{2\mathbf{n}}(\mathbf{n}^2+\mathbf{n})(\mathbf{r}+\mathbf{r}')+\frac{1}{6\,\mathbf{n}^2}(2\,\mathbf{n}^3+3\,\mathbf{n}^2$$
$$+\mathbf{n})\,\mathbf{rr}'\Big].$$

Expanding the parentheses, arranging the terms according to the powers of $\mathbf{n}$, and dividing by $\mathbf{n}$, we obtain for the *mean* of the $\mathbf{n}$ areas —

$$\mathbf{pq}\,\tfrac{1}{6} \sin \lambda \Big\{(6+3\,[\mathbf{r}+\mathbf{r}']+2\,\mathbf{rr}')+\frac{3(\mathbf{r}+\mathbf{r}'+\mathbf{rr}')}{\mathbf{n}}$$
$$+\frac{\mathbf{rr}'}{\mathbf{r}^2}\Big\}.$$

When the points of division coincide with all the points of the

distance $\mathbf{h}$, the number $\mathbf{n}$ of these points becomes infinite, and the mean area of the parallelogram is found to be —

$$\tfrac{1}{6}\,\mathbf{pq}\sin\lambda\,(6+3\,[\mathbf{r}+\mathbf{r}']+2\,\mathbf{rr}');$$

whence the volume is —

$$\tfrac{1}{6}\,\mathbf{pq}\sin\lambda\,(6+3\,[\mathbf{r}+\mathbf{r}']+2\,\mathbf{rr}')\,\mathbf{h}\sin\iota,$$

or, putting H, the perpendicular distance between the first and last positions of the generating plane, in place of $\mathbf{h}\sin\iota$,

$$\text{Volume}=\tfrac{1}{6}\,\mathrm{H}\,\mathbf{pq}\sin\lambda\,(6+3\,[\mathbf{r}+\mathbf{r}']+2\,\mathbf{rr}').$$

The quantity within the parenthesis may be analysed as follows: If, in the formula, area $=\mathbf{pq}\sin\lambda\,(1+\frac{\mathbf{m}}{\mathbf{n}}\mathbf{r})\,(1+\frac{\mathbf{m}}{\mathbf{n}}\mathbf{r}')$, obtained near the beginning of this §, we put $\frac{1}{2}$ for $\frac{\mathbf{m}}{\mathbf{n}}$, and expand the parenthesis, we shall have for the area of the parallelogram *midway* between its first and last positions —

$$\mathbf{pq}\sin\lambda\,(1+\tfrac{1}{2}\,[\mathbf{r}+\mathbf{r}']+\tfrac{1}{4}\,\mathbf{rr}').$$

Four times this area is —

$$\mathbf{pq}\sin\lambda\,(4+2\,[\mathbf{r}+\mathbf{r}']+\mathbf{rr}').$$

Also we have for its area in its first position $\mathbf{pq}\sin\lambda$, and in its last position $\mathbf{pq}\sin\lambda\,(1+[\mathbf{r}+\mathbf{r}]+\mathbf{rr})$.

Now, if A denote the area of the moving figure in the first position, $\mathbf{a}$ the area in the last position, and M the area when midway between the two, it will be found on adding together the above values that —

$$\mathrm{A}+4\,\mathrm{M}+\mathbf{a}=\mathbf{pq}\sin\lambda\,(6+3\,[\mathbf{r}+\mathbf{r}']+2\,\mathbf{rr}').$$

Hence the formula for the volume may be written —

$$\tfrac{1}{6}\,\mathrm{H}\,(\mathrm{A}+4\,\mathrm{M}+\mathbf{a}).$$

183. The formula at the end of § 179 can be analysed in the same way. In that §, the area of the generating figure at the $\mathbf{m}^{\text{th}}$ point of division was found to be $\frac{\mathrm{A}}{\mathbf{n}^2}(\mathbf{n}-\mathbf{r}\,\mathbf{m})^2$. If we put $\frac{\mathbf{n}}{2}$

for $\mathbf{m}$, we find for the area of the figure midway between its extreme positions —

$$M = A\left(1 - \frac{\mathbf{r}}{2}\right)^2;$$

whence $$4\,M = A\,(4 - 4\,\mathbf{r} + \mathbf{r}^2).$$

Also $$\mathbf{a} = A\,(1 - 2\mathbf{r} + \mathbf{r}^2),$$

whence $$A + 4\,M + \mathbf{a} = A\,(6 - 6\,\mathbf{r} + 2\,\mathbf{r}^2),$$

and the formula for the volume may be written —

$$\tfrac{1}{6}\,H\,(A + 4\,M + \mathbf{a}).$$

184. A volume belonging to either of the two species thus far discussed is called in general a *frustum*. A frustum generated in the manner described in § 176 is called a *pyramidal frustum*, if the generating figure be bounded by right lines, but a *conical frustum* when the boundary is a curve. A frustum generated in the manner described in § 182 is called a *prismoid*. The two extreme positions of the generating figure, between which the volume is contained, are called the *bases* of the frustum. Any other position of the generating figure is a *principal section*. The *perpendicular* distance between the two bases is the *altitude*. The formula of §§ 182, 183 afford the following general rule for the mensuration of any kind of frustum: —

The numerical measure of the frustum is found by multiplying one-sixth of the altitude into the sum of the areas of the two bases, and of four times the principal section midway between them.

185. When two of the lines described by the vertices of the parallelogram which generates a prismoid (§ 182) intersect in the plane of one of the bases, that side of the base which is included between these vertices vanishes, and the base itself, instead of a parallelogram, becomes a portion of a right line, whose area of course is zero. The prismoid is in this case called a *wedge*, and the rectilinear distance which represents one of the bases is the *edge*. The *length* of the remaining base is the length of either of its two sides which are parallel to the edge, and the perpendicular distance between those two sides is the *breadth* of the base. In

the notation of § 182, if $\mathbf{p}$ represent the length of the base, then then $\mathbf{q}$ sine λ will be the breadth, the edge will be $\mathbf{p}\ (1 + \mathbf{r})$, and the side $\mathbf{q}\ (1 + \mathbf{r}')$ will vanish; that is —

$$(1 + \mathbf{r}') = 0 \qquad \text{whence} \qquad \mathbf{r}' = -1.$$

Applying this value of $\mathbf{r}'$ to the formula for the volume near the end of the § —

$$\text{Volume} = \tfrac{1}{6}\,\mathrm{H}\,\mathbf{pq}\,\sin\lambda\,(6 + 3\,[\mathbf{r} + \mathbf{r}'] + 2\,\mathbf{rr}'),$$

the latter becomes —

$$\tfrac{1}{6}\,\mathrm{H}\,\mathbf{pq}\,\sin\lambda\,(6 + 3\,[\mathbf{r} - 1] - 2\,\mathbf{r}),$$

or

$$\tfrac{1}{6}\,\mathrm{H}\,\mathbf{pq}\,\sin\lambda\,(3 + \mathbf{r}),$$

which may be written —

$$\tfrac{1}{6}\,\mathrm{H}\,.\,\mathbf{q}\,\sin\lambda\,(2\,\mathbf{p} + \mathbf{p}\,[1 + \mathbf{r}]).$$

Whence the rule —

The measure of the wedge is obtained by adding the edge to twice the length of the base, and multiplying the sum into one-sixth the product of the breadth of the base and the altitude.

186. The wedge is plainly to be regarded as a *limiting case* of the prismoid, characterized by the fact that the area of one of the bases disappears. The other species of frustum has its corresponding limiting case when the apex is situated at the extremity of the directrix. One of the bases then becomes a *point*, and its area $\mathbf{a}$ is 0. In this case the pyramidal frustum becomes a *pyramid*, and the conical frustum a *cone*, though the latter name is also applied to the *surface* described by the periphery of the generating figure. The measure of the volume is most readily found by the formula of § 180, when, as $\mathbf{a} = 0$, $\sqrt{\mathrm{A}\mathbf{a}} = 0$, and the formula becomes $\tfrac{1}{3}\,\mathrm{A}\mathbf{h}\,\sin\iota$, or $\tfrac{1}{3}\,\mathrm{A}\mathbf{h}$.

The content of a pyramid or cone is one-third the product of the base and altitude.

187. Another limiting case of the frustum occurs when r is put equal to 0 in the formula of § 179, or when both $\mathbf{r} = 0$ and $\mathbf{r}' = 0$ in § 182. The effect is, in either case, to make all the principal

sections equal to each other. Thus, in § 179, the formula for the area of a principal section is $\frac{A}{\mathbf{n}^2}(\mathbf{n} - \mathbf{rm})^2$, which becomes A when $\mathbf{r} = 0$, and in § 182 the area of a principal section is —

$$\mathbf{pq} \sin \lambda \left(1 + \frac{\mathbf{m}}{\mathbf{n}}\mathbf{r}\right)\left(1 + \frac{\mathbf{m}}{\mathbf{n}}\mathbf{r}'\right);$$

and this, when $\mathbf{r} = 0$ and $\mathbf{r}' = 0$, becomes $\mathbf{pq} \sin \lambda$ or A. In this case the conical frustum becomes a *cylinder* (another name used for the surface as well as the volume), and a frustum generated by a polygon becomes a *prism.* When the generating polygon is a parallelogram (as in the prismoid), a variety of the prism, called a *parallelopiped*, is produced. A parallelopiped whose bases are square, whose directrix is perpendicular to the plane of the base, and whose altitude is equal to the side of the base, is called a *cube.*

As all the principal sections, and the two bases are equal, the content is readily found from either § 180, § 182, or § 183, for in all these formulæ, when $\mathbf{a} = M = A$, the expression for the content becomes —

$$A\mathbf{h} \sin \iota \qquad \text{or} \qquad AH.$$

And this again agrees with the formula of § 175, for the generating figure has in this case a *constant* area.

The content of any prism or cylinder is the product of its base by its altitude.

The content of the cube is the third power of the side of its base, whence the use of the name "cube" as a synonyme for "third power."

188. A conical frustum generated by the motion of a circle along a directrix which passes through its centre and is perpendicular to its plane, is a *right circular conical frustum;* and in the two limiting cases just noticed becomes a *right circular cone*, or a *right circular cylinder.* These three volumes are of special importance as belonging to the class of *volumes of revolution*, — so called because they may be generated not only in the manner already described, when the generatrix is a right line, but also upon a *circular directrix.* Thus, in Fig. 50, it is apparent that the frustum may be generated not only by the motion of the circle C

along the directrix $O\mathbf{o}$, but by the revolution of the trapezoid T around its side $\overline{O\mathbf{o}}$ as an *axis;* in which case every point of the revolving figure describes a circle, and one of these circles may be taken as the directrix.

189. In all volumes of revolution, the generating figure is a plane figure of constant area and form. In the right circular conical frustum, this figure is a trapezoid, whose parallel sides are perpendicular to a third side, the axis of revolution. In the limiting cases, this trapezoid becomes a right-angled triangle for the cone, or a rectangle for the cylinder. That side of the generating figure which is not perpendicular to the axis is called the *side* of the frustum, cone, or cylinder. In revolving, the side describes a *conical* or *cylindrical surface.* The remainder of the surface which bounds the volume consists of plane figures (§ 167), — the circular *bases* of the volume.

190. If the radii of the bases of the frustum be R and $\mathbf{r}$, then in the formula of § 180 —

$$A = \pi R^2 \qquad \text{and} \qquad \mathbf{a} = \pi \mathbf{r}^2 \ (\S\ 145),$$

whence $$\sqrt{A\mathbf{a}} = \pi R\mathbf{r}.$$

Hence the content of a right circular frustum is —

$$\textit{Volume} = \tfrac{1}{3} \pi H (R^2 + R\mathbf{r} + \mathbf{r}^2).$$

When this frustum becomes a cone $\mathbf{r} = 0$, and when it becomes a cylinder $\mathbf{r} = R$; so that —

The content of a right circular cone is $\frac{1}{3} \pi R^2H$; and of a right circular cylinder, πR^2H.

191. Expressions in terms of H, $\mathbf{r}$, and R may also be obtained for the conical or cylindrical *surfaces* which bound these volumes; and the process will be quite similar, whichever mode of generation be adopted. If we regard the circle as the generatrix, and take for the directrix a right line drawn from any point of the curve to the apex, we must find the mean length of the moving periphery, multiply it by the distance over which it moves, and this product by the sine of the angle which the curved generatrix makes with the rectilinear directrix. Again, if the curve be taken as the directrix, that which was formerly the directrix becomes the

generatrix, and the length of the directrix is different for different points of the generatrix. It is therefore necessary to find the mean length of the directrix, multiply it by the length of the generatrix, and this product by the sine of the angle between them. So that the three quantities whose product is to be taken are the same in either case, — viz: the side of the frustum, the mean circumference of all principal sections, and the sine of the angle which the circumference of such a section makes with the side.

192. *First. — To find the side of the frustum.* If the radii of the bases are R and $\mathbf{r}$, the altitude H, and the side S, the student will readily show that —

$$S = \sqrt{H^2 + (R - \mathbf{r})^2}.$$

In the cone —

$$\mathbf{r} = 0 \;\therefore\; S = \sqrt{H^2 + R^2}.$$

In the cylinder —

$$\mathbf{r} = R \;\therefore\; S = H.$$

193. *Second. — To find the mean circumference of principal sections.* Any such circumference is the product of its radius by 2π (§ 145), and its radius is the perpendicular distance of a point on the side of the frustum from its axis. Hence the mean distance of all such points is to be found, and multiplied by 2π. If for this purpose a number of points be taken on the side, and perpendiculars to the axis be drawn through them, the axis will also be divided into equal parts (§ 37). Hence the mean distance of the points of the side from the axis is the same as the mean ordinate of the side (§ 115), which is the ordinate of its middle point, or the mean of the ordinates of its extremities; that is, $\frac{1}{2}(R + \mathbf{r})$. Hence the mean circumference sought is —

$$2\pi \times \tfrac{1}{2}(R + \mathbf{r}) \qquad \text{or} \qquad \pi(R + \mathbf{r}).$$

In the case of the cone this becomes πR; in that of the cylinder, $2\pi R$.

[It is to be noted that the mean distance of the points of a *curve* from an axis is not in general the same as the mean ordinate of the curve, for the perpendiculars in the former case are drawn so as to cut off equal lengths on the curve, and in the latter case on the axis. But § 37 shows that *for the right line* the mean distance and mean ordinate are the same.]

194. *Third.* — *To find the sine of the angle between the side of the frustum and the circumference of a principal section.* (This angle is not the same as the inclination of the side to the *plane* of the section; for, as was seen in § 157, it is possible, in a plane having a given inclination to a given line, to draw right lines making very different angles with it, and the same is true of curves.) Let the circumference be divided into n equal parts, and let lines (A M, A N, Fig. 50) extend from the apex to two successive points of division. These two points are at equal distances from the apex, hence the angles, which the two lines make with the line MN which joins their extremities, are equal (§ 41, Ex. 15); and the sum of these two angles, together with the angle k, which the two lines make with each other, is equal to 180°. Hence the line M N makes with the side of the frustum an angle of $\frac{180° - k}{2}$. The greater the value of **n**, the more nearly will M N coincide with the arc of the circle, but when **n** is infinitely great k is zero, hence the angle which the circumference makes with the side of the frustum is $\frac{180°}{2}$, or 90°. As the sine of this angle is 1, to multiply by it will not affect the product of the other two factors, hence the area sought is the product of the side of the frustum into the mean circumference of principal sections.

195. The formula for the area of the conical surface of the frustum is therefore —

$$\pi S (R + \mathbf{r}) \quad \text{or} \quad \pi (R + \mathbf{r}) \sqrt{H^2 + (R - \mathbf{r})^2}.$$

In the case of the cone, since $\mathbf{r} = 0$, the above formula becomes —

$$\pi R S \quad \text{or} \quad \pi R \sqrt{H^2 + R^2};$$

and in the case of the cylinder, where —

$$\mathbf{r} = R \qquad \text{cylindrical surface} = 2 \pi R H.$$

In general, whenever one of two right lines which are in the same plane remains fixed in position, and the other revolves about it, so that every point of the revolving line describes a circle, the area of the surface generated by any distance on the revolving line is the product of that

distance into the circumference described by its middle points.

The student may show that this rule applies when the lines are perpendicular, and will notice that a plane (§ 167) is a variety of conical surface.

196. A *sphere* is a volume generated by the revolution of a semi-circle about its diameter as an axis.

The distance of all the points of the surface of a sphere from the middle point of this axis (called the *centre* of the sphere), are equal; and any one of these equal distances is named a *radius* of the sphere. That part of the spherical surface described by any arc of the generating circle is a *zone*, and the volume generated by a figure bounded by an arc and two radii is called a *spherical sector.*

197. The surface and contents of the sphere may be computed by first measuring the area and volume produced by the revolution of a triangle whose sides are a *chord* of the generating figure and radii drawn to its extremities. Let the semi-circular curve be divided into n equal parts, and let C (in Fig. 51) be the $\mathbf{m}^{\text{th}}$ point of division, while B is the $(\mathbf{m}-1)^{\text{th}}$ point. Then if O be the centre, and A O H the axis, about which the whole figure revolves, we have first to find the surface S, which will be generated by the revolution of the distance $\overline{BC}$, and the volume $\mathbf{v}$ generated by the triangle bounded by $\overline{BC}$ with $\overline{OB}$ and $\overline{OC}$. Let the angle made with the axis by OC be called 2α, and that made by OB be 2β, then, by § 144—

$$2\alpha = \frac{\mathbf{m}}{\mathbf{n}}\,180° \qquad \text{and} \qquad 2\beta = \frac{\mathbf{m}-1}{\mathbf{n}}\,180°. \qquad (1)$$

Let $\mathbf{r}$ denote the radius $\overline{OA}$, $\overline{OB}$, etc. The line OC makes with OB an angle of $2\alpha - 2\beta$. If a line OD be drawn from O equally dividing this angle, or making with OB the angle $\alpha - \beta$, its angle with OA will be $2\beta + (\alpha - \beta)$, or $\alpha + \beta$. Such a line (§ 141, Ex. 16) will meet $\overline{BC}$ perpendicularly at its middle point D. From the points B, C, D, let lines pass perpendicular to the axis; and let the expression, "the volume under a given distance," be understood to mean the volume generated by the revolution of a figure bounded by the axis, the given distance, and perpendiculars

to the former joining it with the extremities of the latter. The following are the values of various distances on the figure, which the student will verify:—

$$\overline{BC} = 2\,\overline{BD} = 2\,\mathbf{r}\sin(\alpha - \beta) \quad \text{and} \quad \overline{OD} = \mathbf{r}\cos(\alpha - \beta);$$

$$\overline{KD} = \overline{OD}\sin(\alpha + \beta) = \mathbf{r}\cos(\alpha - \beta)\sin(\alpha + \beta);$$

$$\overline{FB} = \mathbf{r}\sin 2\beta \quad \text{and} \quad \overline{OF} = \mathbf{r}\cos 2\beta;$$

$$\overline{GC} = \mathbf{r}\sin 2\alpha \quad \text{and} \quad \overline{OG} = \mathbf{r}\cos 2\alpha;$$

$$\overline{GF} = \mathbf{r}(\cos 2\beta - \cos 2\alpha).$$

By the rule of § 195, the surface described by $\overline{BC}$ is equal to $2\pi\,\overline{KD}\;\overline{BC}$; whence—

$$S = 4\pi\mathbf{r}^2\cos(\alpha - \beta)\sin(\alpha + \beta)\sin(\alpha - \beta). \qquad (2)$$

The factor $\sin(\alpha + \beta)\sin(\alpha - \beta)$ may be simplified by first expanding, by § 101, to the form—

$$(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\ (\sin\alpha\cos\beta - \cos\alpha\sin\beta);$$

or

$$\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta;$$

which expression, when $1 - \sin^2\alpha$ is put for $\cos^2\alpha$, and $1 - \sin^2\beta$ for $\cos^2\beta$, reduces to the form $\sin^2\alpha - \sin^2\beta$.

$$\therefore \quad \sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta. \qquad (3)$$

Whence, substituting in (2),

$$S = 4\pi\mathbf{r}^2\cos(\alpha - \beta)\ (\sin^2\alpha - \sin^2\beta). \qquad (4)$$

The volume $\mathbf{v}$ may be found by adding together the volumes under $\overline{BC}$ and $\overline{OC}$ and subtracting that under $\overline{OB}$. For these we have by § 190—

$$\text{Volume under } \overline{BC} = \tfrac{1}{3}\pi\,\overline{GF}\,.\,(\overline{GC}^2 + \overline{GC}\,.\,\overline{FB} + \overline{FB}^2).$$

$$= \tfrac{1}{3}\pi\mathbf{r}^3(\cos 2\beta - \cos 2\alpha)\ (\sin^2 2\alpha + \sin 2\alpha\sin 2\beta + \sin^2 2\beta)$$

$$= \tfrac{1}{3}\pi\mathbf{r}^3\cos 2\beta\,(\sin^2 2\alpha + \sin 2\alpha\sin 2\beta + \sin^2 2\beta) - \tfrac{1}{3}\pi\mathbf{r}^3\cos 2\alpha\,(\sin^2 2\alpha + \sin 2\alpha\sin 2\beta + \sin^2 2\beta).$$

$$\text{Volume under } \overline{OC} = \tfrac{1}{3}\pi\,\overline{OG}\,.\,\overline{GC}^2 = \tfrac{1}{3}\pi\mathbf{r}^3\cos 2\alpha\sin^2 2\alpha.$$

$$\text{Volume under } \overline{OB} = \tfrac{1}{3}\pi\,\overline{OF}\,.\,\overline{FB}^2 = \tfrac{1}{3}\pi\mathbf{r}^3\cos 2\beta\sin^2 2\beta.$$

$$\therefore \quad \mathbf{v} = \tfrac{1}{3}\pi\mathbf{r}^3\cos 2\beta\,(\sin^2 2\alpha + \sin 2\alpha\sin 2\beta) - \tfrac{1}{3}\pi\mathbf{r}^3\cos 2\alpha\,(\sin 2\alpha\sin 2\beta + \sin^2 2\beta).$$

$= \frac{1}{3} \pi \mathbf{r}^3 \sin 2\alpha \cos 2\beta (\sin 2\alpha + \sin 2\beta) - \frac{1}{3} \pi \mathbf{r}^3 \cos 2\alpha \sin 2\beta$
$(\sin 2\alpha + \sin 2\beta)$.

$= \frac{1}{3} \pi \mathbf{r}^3 (\sin 2\alpha + \sin 2\beta) \quad (\sin 2\alpha \cos 2\beta - \cos 2\alpha \sin 2\beta)$.

Wherefore, from § 110—

$$\mathbf{v} = \frac{1}{3} \pi \mathbf{r}^3 (\sin 2\alpha + \sin 2\beta) \sin (2\alpha - 2\beta). \qquad (5)$$

Now it was found in § 134, equation (1), that when β and γ are any two angles—

$$\sin \beta + \sin \gamma = 2 \sin \tfrac{1}{2} (\beta + \gamma) \cos \tfrac{1}{2} (\beta - \gamma).$$

If 2α be put for the β of this formula, and 2β for γ, the formula becomes—

$$\sin 2\alpha + \sin 2\beta = 2 \sin (\alpha + \beta) \cos (\alpha - \beta). \qquad (6)$$

Moreover, by § 102— $\sin 2\alpha = 2 \sin \alpha \cos \alpha$,

whence—

$$\sin (2\alpha - 2\beta) = \sin 2(\alpha - \beta) = 2 \sin (\alpha - \beta) \cos (\alpha - \beta). \quad (7)$$

Substituting from (6) and (7) in (5), we have—

$$\mathbf{v} = \frac{1}{3} \pi \mathbf{r}^3 \,.\, 2 \sin (\alpha + \beta) \cos (\alpha - \beta) \,.\, 2 \sin (\alpha - \beta) \cos (\alpha - \beta),$$

or $\quad \mathbf{v} = \frac{4}{3} \pi \mathbf{r}^3 \cos^2 (\alpha - \beta) \sin (\alpha + \beta) \sin (\alpha - \beta). \qquad (8)$

Substituting in (8) the value of $\sin (\alpha + \beta) \sin (\alpha - \beta)$ derived from (3), we have—

$$\mathbf{v} = \frac{4}{3} \pi \mathbf{r}^3 \cos^2 (\alpha - \beta) \quad (\sin^2 \alpha - \sin^2 \beta). \qquad (9)$$

198. The next step is to find the aggregate surface and volume generated by a number of contiguous chords or triangles. For this purpose we return to the values of α and β as found from equation (1) of the preceding §, viz:—

$$\alpha = \frac{\mathbf{m}}{2\mathbf{n}} 180° \qquad \text{and} \qquad \beta = \frac{\mathbf{m} - 1}{2\mathbf{n}} 180°,$$

whence $\quad \alpha - \beta = \dfrac{1}{2\mathbf{n}} 180°.$

Substituting these values in equations (4) and (9) we have—

$$S = 4 \pi \mathbf{r}^2 \cos \frac{1}{2\mathbf{n}} 180° \left(\sin^2 \frac{\mathbf{m}}{2\mathbf{n}} 180° - \sin^2 \frac{\mathbf{m} - 1}{2\mathbf{n}} 180° \right);$$

and $\quad \mathbf{v} = \frac{4}{3} \pi \mathbf{r}^3 \cos^2 \dfrac{1}{2\mathbf{n}} 180° \left(\sin^2 \dfrac{\mathbf{m}}{2\mathbf{n}} 180° - \sin^2 \dfrac{\mathbf{m} - 1}{2\mathbf{n}} 180° \right).$

Replacing $\mathbf{m}$ by the numbers 1, 2, 3, 4, etc., successively, we have:
When —

$$\mathbf{m}=1,\ \mathbf{s}=4\,\pi\,\mathbf{r}^2\cos\frac{1}{2\,\mathbf{n}}\,180^\circ\left(\sin^2\frac{1}{2\,\mathbf{n}}\,180^\circ-\sin^2 0^\circ\right),$$

when —

$$\mathbf{m}=2,\ \mathbf{s}=4\,\pi\,\mathbf{r}^2\cos\frac{1}{2\,\mathbf{n}}\,180^\circ\left(\sin^2\frac{2}{2\,\mathbf{n}}\,180^\circ-\sin^2\frac{1}{2\,\mathbf{n}}\,180^\circ\right),$$

when —

$$\mathbf{m}=3,\ \mathbf{s}=4\,\pi\,\mathbf{r}^2\cos\frac{1}{2\,\mathbf{n}}\,180^\circ\left(\sin^2\frac{3}{2\,\mathbf{n}}\,180^\circ-\sin^2\frac{2}{2\,\mathbf{n}}\,180^\circ\right),$$

etc., etc.

It is obvious that when such a series of quantities is added, the common factor $4\,\pi\,\mathbf{r}^2\cos\frac{1}{2\,\mathbf{n}}\,180^\circ$ will be multiplied in the result by a parenthesis consisting of only two terms, all the other (intermediate) terms being cancelled in the addition, so that the sum of the surfaces described by $\mathbf{p}$ successive chords, the first of which is adjacent to the axis, is —

$$4\,\pi\,\mathbf{r}^2\cos\frac{1}{2\,\mathbf{n}}\,180^\circ\left(\sin^2\frac{\mathbf{p}}{2\,\mathbf{n}}\,180^\circ-\sin^2 0^\circ\right);$$

or, if δ be put for the angle $\frac{\mathbf{p}}{\mathbf{n}}\,180^\circ$ which the radius extending to the $\mathbf{p}^{\text{th}}$ point of division makes with the axis, we shall have, — remembering that $\sin^2\frac{1}{2}\,\delta=\frac{1}{2}\,(1-\cos\delta)$, (§ 104), and that $\sin 0^\circ = 0$, (§ 105): —

$$\text{sum of } \mathbf{p} \text{ terms} = 2\,\pi\,\mathbf{r}^2\cos\frac{1}{2\,\mathbf{n}}\,180^\circ\,(1-\cos\delta). \qquad (10)$$

Similarly, if γ denote the angle $\frac{\mathbf{q}}{\mathbf{n}}\,180^\circ$ made with the axis by a radius extending to the $\mathbf{q}^{\text{th}}$ point of division, the sum of the surfaces described by $\mathbf{q}$ successive chords, the first of which is adjacent to the axis, is —

$$\textit{sum of } \mathbf{q} \textit{ terms} = 2\,\pi\,\mathbf{r}^2\cos\frac{1}{2\,\mathbf{n}}\,180^\circ\,(1-\cos\gamma). \qquad (11)$$

Subtracting (11) from (10), we have for the aggregate surface

described by the chords extending from the $\mathbf{q}^{th}$ to the $\mathbf{p}^{th}$ point of division —

$$\textit{sum of } (\mathbf{p} - \mathbf{q}) \textit{ terms } 2\,\pi\,\mathbf{r}^2 \cos\frac{1}{2\,\mathbf{n}}\,180\ (\cos\gamma - \cos\delta). \qquad (12)$$

If in (10) we let $\mathbf{p} = \mathbf{n}$; that is, if the successive chords be extended along the entire curve of the semi-circle, then the angle δ or $\frac{\mathbf{p}}{\mathbf{n}}\,180°$, becomes 180°, and $\cos\delta$ becomes -1 (§ 105); whence the entire surface described by all the chords of the semi-perimeter is —

$$\textit{surface} = 4\,\pi\,\mathbf{r}^2 \cos\frac{1}{2\,\mathbf{n}}\,180°. \qquad (13)$$

Proceeding with the formula for $\mathbf{v}$ in Equation (9) in precisely the same manner as above with $\mathbf{s}$, we shall find that the aggregate volume described by the triangles whose vertices lie between the $\mathbf{q}^{th}$ and $\mathbf{p}^{th}$ point of division is —

$$\textit{sum of } (\mathbf{p} - \mathbf{q}) \textit{ terms} = \tfrac{2}{3}\,\pi\,\mathbf{r}^3 \cos\frac{1}{2\,\mathbf{n}}\,180°\ (\cos\gamma - \cos\delta), \qquad (14)$$

And that the total volume generated by the revolution of the semi-polygon is —

$$\textit{volume} = \tfrac{4}{3}\,\pi\,\mathbf{r}^3 \cos\frac{1}{2\,\mathbf{n}}\,180°. \qquad (15)$$

199. If now the series of chords be brought into coincidence with the curve by making their number $\mathbf{n}$ infinite, then $\cos\frac{1}{2\,\mathbf{n}}\,180°$ becomes cos 0° or 1. On substituting this value in equations (13) and (15) of § 198, the formulæ for the surface and content of the sphere are obtained. They are —

$$\textit{surface of sphere} = 4\,\pi\,\mathbf{r}^2,$$
$$\textit{content of sphere} = \tfrac{4}{3}\,\pi\,\mathbf{r}^3.$$

By comparison with the formula of § 145 these results may be stated in words as follows: —

The surface of the sphere is four times the area of a circle of equal radius; and —

The content of the sphere is the product of its surface by one-third its radius.

200. The same substitution, of 0° for $\frac{1}{2\,\mathbf{n}}\,180°$ reduces equations (12) and (14) of § 198 to formulæ for the area of a zone and the content of a spherical sector. These formulæ are —

$$\textit{area of a zone} = 2\,\pi\,\mathbf{r}^2\,(\cos\gamma - \cos\delta)\text{; and}$$
$$\textit{spherical sector} = \tfrac{2}{3}\,\pi\,\mathbf{r}^3\,(\cos\gamma - \cos\delta).$$

The formula for the zone may be factored, and written thus —

$$2\,\pi\,\mathbf{r}\,(\mathbf{r}\cos\gamma - \mathbf{r}\cos\delta).$$

In Fig. 51, if the angle of OB with the axis be called γ, and that of OC be δ, then the formula represents the area of the zone described by the revolution of the arc $\overline{BC}$. The factor $2\,\pi\,\mathbf{r}$ is the periphery of a circle whose radius is $\mathbf{r}$ (§ 145), while the factor $(\mathbf{r}\cos\gamma - \mathbf{r}\cos\delta)$ represents the distance $\overline{OF} - \overline{OG}$ or $\overline{GF}$, the distance cut off upon the axis between perpendiculars from the extremities of the generating arc $\overline{BC}$. This distance is called the *altitude* of the zone. Hence —

The area of a zone is the product of its altitude by the periphery of a circle whose radius is that of the sphere; and —

The content of a spherical sector is the product of the area of its zone by one-third the radius of the sphere.

Examples.

1. The pyramid of Cheops was built on a square base, each of the sides of which was 764 feet long. The summit, vertically over the centre of the base, had originally a height of 479 feet. Required the area of the base and of each face, and the volume of masonry composing the structure.

2. A cubic foot of water weighs a thousand ounces; required how many pounds are contained in a tank six feet long and four feet broad, when it is filled to a depth of three feet.

3. Required the surface and weight of an upright hexagonal block of Quincy granite (specific gravity 2.652) each side of which measures 18 inches and the height is 3 feet.

4. What is the weight of a cylindrical column of the same material, 4 feet long and 15 inches in diameter?

5. A certain cask is composed of two equal conical frustums joined at their larger bases. The largest diameter is 28 inches, the diameter of the head 20 inches, and the length 40 inches; how many gallons of wine will it hold, there being 231 cubic inches in a gallon?

6. What must be the diameter of a cylindrical quart cup, 6 inches deep?

7. How many cubic feet of earth are contained in an embankment 10 feet high, having a level top 20 feet wide and 40 feet long, and resting on a level base 40 feet wide and 60 feet long?

8. Find the surface and content of a cone whose base is a circle having a circumference of 1 foot 10 inches, and whose altitude is 12 inches.

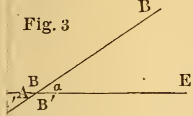
9. At the mean distance of the earth from the sun (which is supposed to be about 93,000,000 miles), his apparent semi-diameter is $16' 01''.5$. Required the diameter and circumference of the sun in miles, his volume, and the area of his surface.

10. On a school-globe, 1 foot in diameter, the tropics are drawn at a latitude of $23\frac{1}{2}°$ and the polar circles at a latitude of $66\frac{1}{2}°$. What is the area of each zone bounded by these circles?

11. Within a right circular cylinder (Fig. 52), whose altitude is equal to the diameter of its base, is inscribed a sphere which touches the two bases and the cylindrical surface. What is the ratio of the surface of the sphere to the entire surface of the cylinder? What is the ratio of their contents?

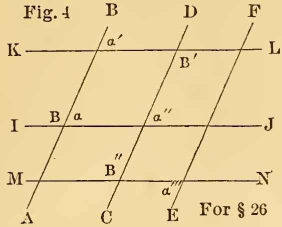
[The student will find the two ratios equal, each being 2:3. This relation between the two volumes is said to have been discovered by Archimedes, who was so much interested in the investigation that he directed it to be commemorated in the device upon his gravestone. Cicero, in his Tusculan Disputations, relates that by this mark he discovered the neglected grave of the mathematician, more than a century after his death.]

Fig. 3



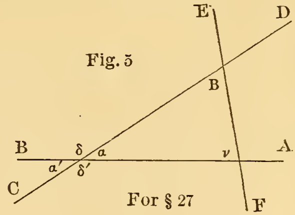
For §§ 22-24.

Fig. 4



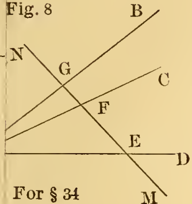
For § 26

Fig. 5



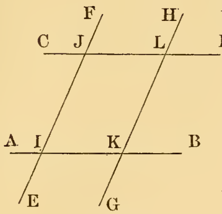
For § 27

Fig. 8



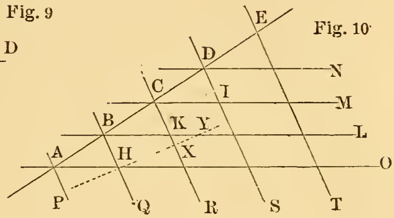
For § 34

Fig. 9



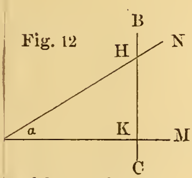
For § 35

Fig. 10



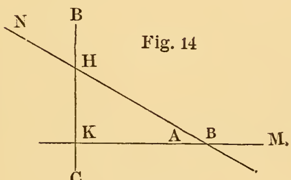
For § 36

Fig. 12



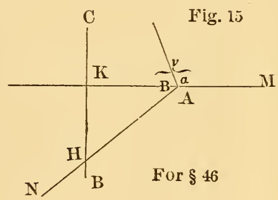
For §§ 41-45 & 49-51

Fig. 14

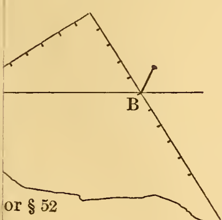


For § 46

Fig. 15

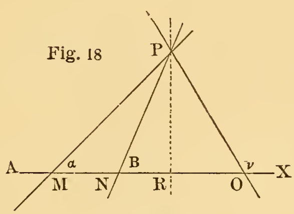


For § 46



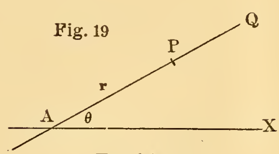
For § 52

Fig. 18



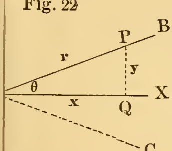
For § 53

Fig. 19



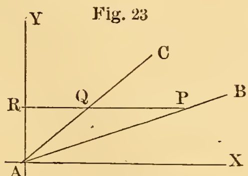
For §§ 55-56

Fig. 22



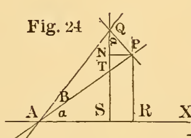
§ 72-82 & 85

Fig. 23



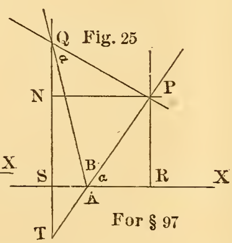
For § 96

Fig. 24



For § 97

Fig. 25



For § 97

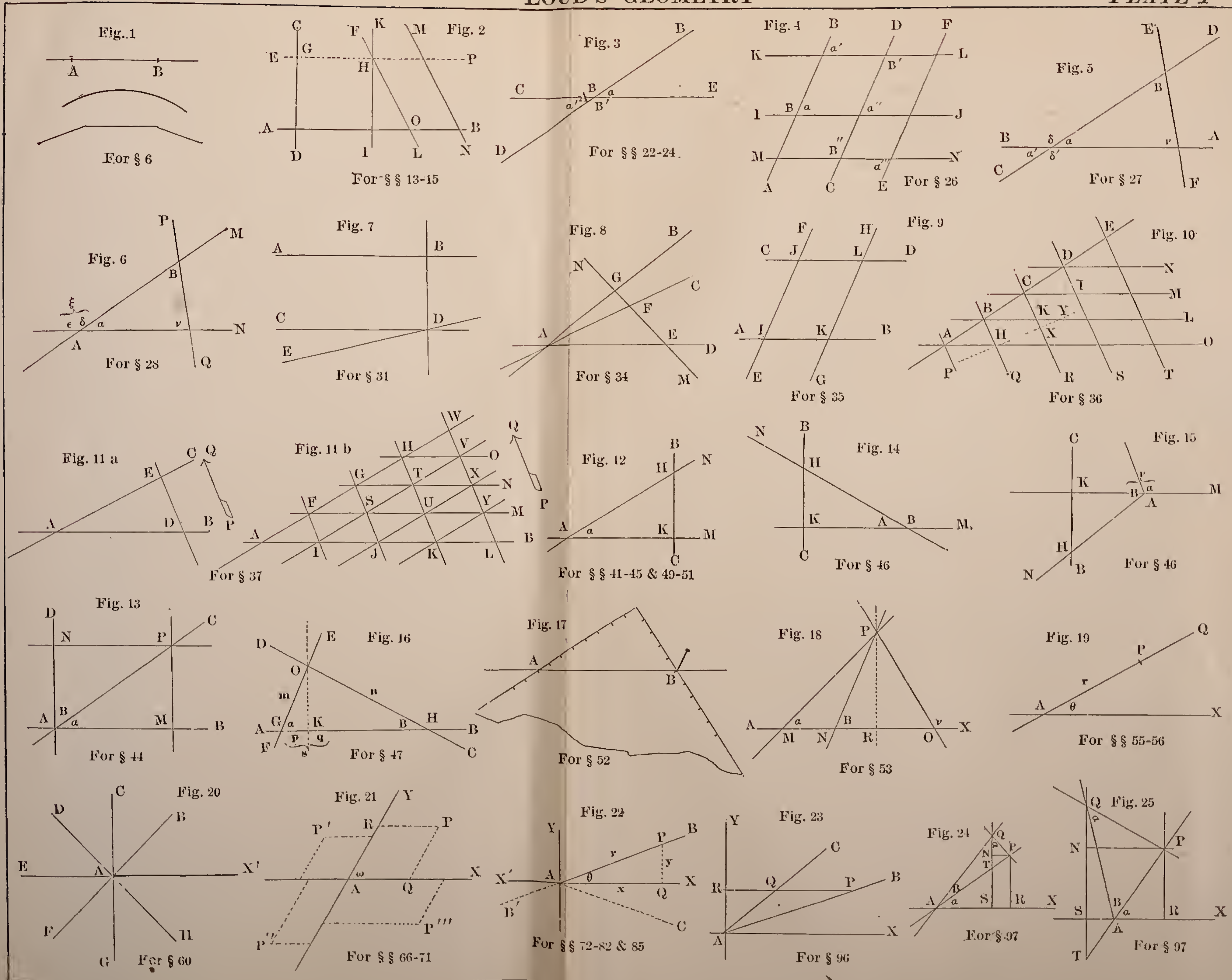


Fig. 1 For § 6

Fig. 2 For §§ 13-15

Fig. 3 For §§ 22-24

Fig. 4 For § 26

Fig. 5 For § 27

Fig. 6 For § 28

Fig. 7 For § 31

Fig. 8 For § 34

Fig. 9 For § 35

Fig. 10 For § 36

Fig. 11 a

Fig. 11 b For § 37

Fig. 12 For §§ 41-45 & 49-51

Fig. 14 For § 46

Fig. 15 For § 46

Fig. 13 For § 44

Fig. 16 For § 47

Fig. 17 For § 52

Fig. 18 For § 53

Fig. 19 For §§ 55-56

Fig. 20 For § 60

Fig. 21 For §§ 66-71

Fig. 22 For §§ 72-82 & 85

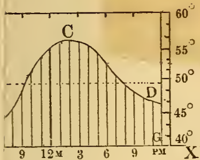
Fig. 23 For § 96

Fig. 24 For § 97

Fig. 25 For § 97

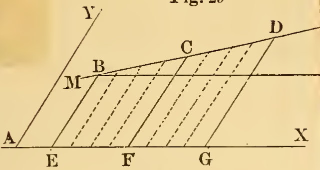
D'S GEOMETRY

Fig. 28



For § 114

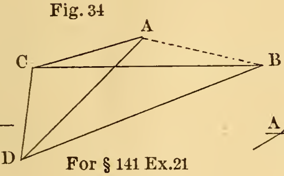
Fig. 29



For § 115

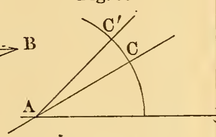
x.2

Fig. 34



For § 141 Ex.21

Fig. 35



For § 144

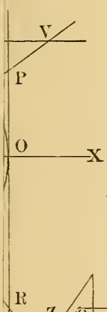
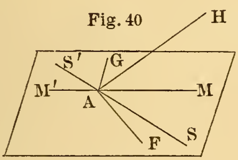
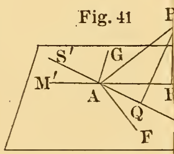


Fig. 40



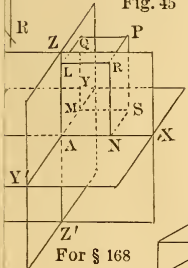
For § 155

Fig. 41



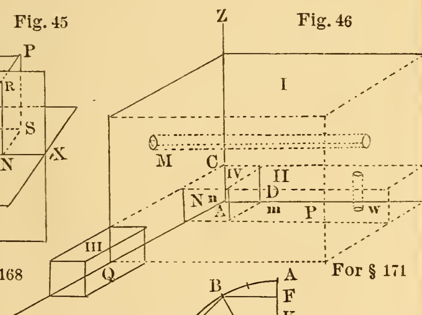
For § 158

Fig. 45



For § 168

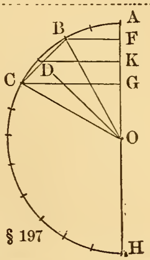
Fig. 46



For § 171

For §§ 188-194

Fig. 51



For § 197

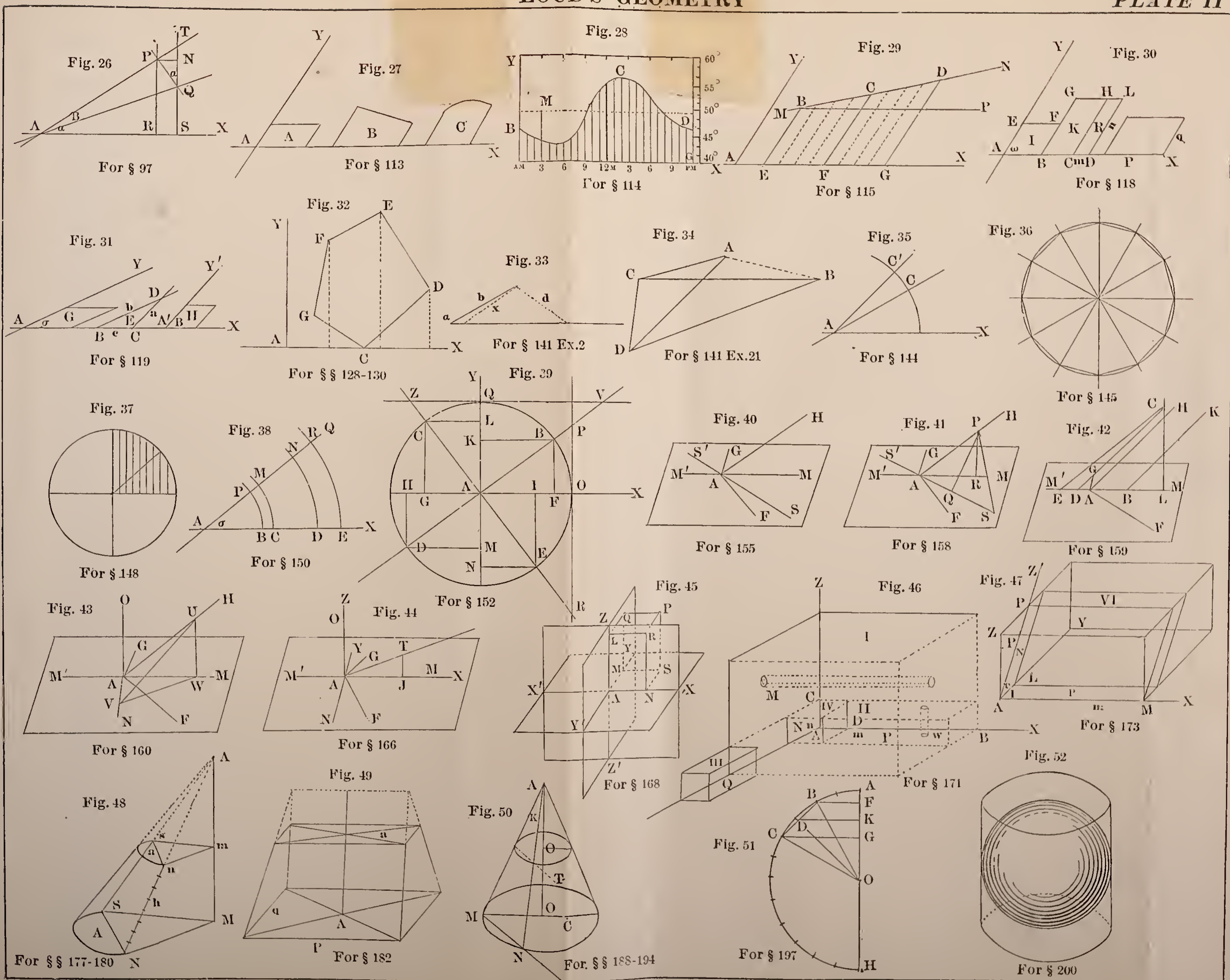


Fig. 26 For § 97

Fig. 27 For § 113

Fig. 28 For § 114

Fig. 29 For § 115

Fig. 30 For § 118

Fig. 31 For § 119

Fig. 32 For §§ 128-130

Fig. 33 For § 141 Ex. 2

Fig. 34 For § 141 Ex. 21

Fig. 35 For § 144

Fig. 36 For § 145

Fig. 37 For § 148

Fig. 38 For § 150

Fig. 39 For § 152

Fig. 40 For § 155

Fig. 41 For § 158

Fig. 42 For § 159

Fig. 43 For § 160

Fig. 44 For § 166

Fig. 45 For § 168

Fig. 46 For § 171

Fig. 47 For § 173

Fig. 48 For §§ 177-180

Fig. 49 For § 182

Fig. 50 For §§ 188-194

Fig. 51 For § 197

Fig. 52 For § 200